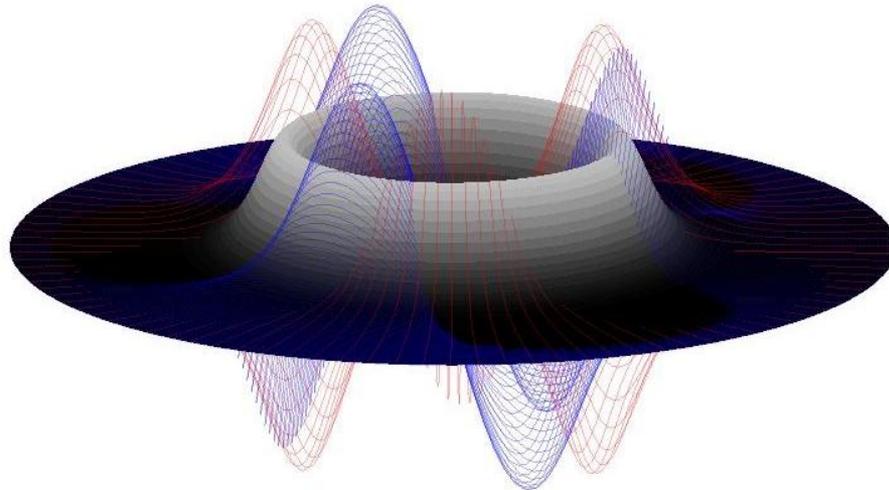


Azimuthal Modulation Stability for Vortices in the Focusing Nonlinear Schrodinger Equation



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Ronald M. Caplan
Ricardo Carretero
Enam Hoq
Panayotis G. Kevrekidis

Overview

- ▣ Introduction and Motivation
- ▣ Local Theory
 - ▣ Formulation of Quasi-1D Problem
 - ▣ Stability Analysis
 - ▣ Bifurcations and Nonlinear Effects
- ▣ Finding Steady-State Solutions
 - ▣ Variational Approach
 - ▣ Numeric Optimization
- ▣ Full System Simulation
 - ▣ Analytic and Numeric Predictions
 - ▣ Numeric Method
 - ▣ Numeric Results
- ▣ Nonlocal Theory
- ▣ Conclusion and Further Research

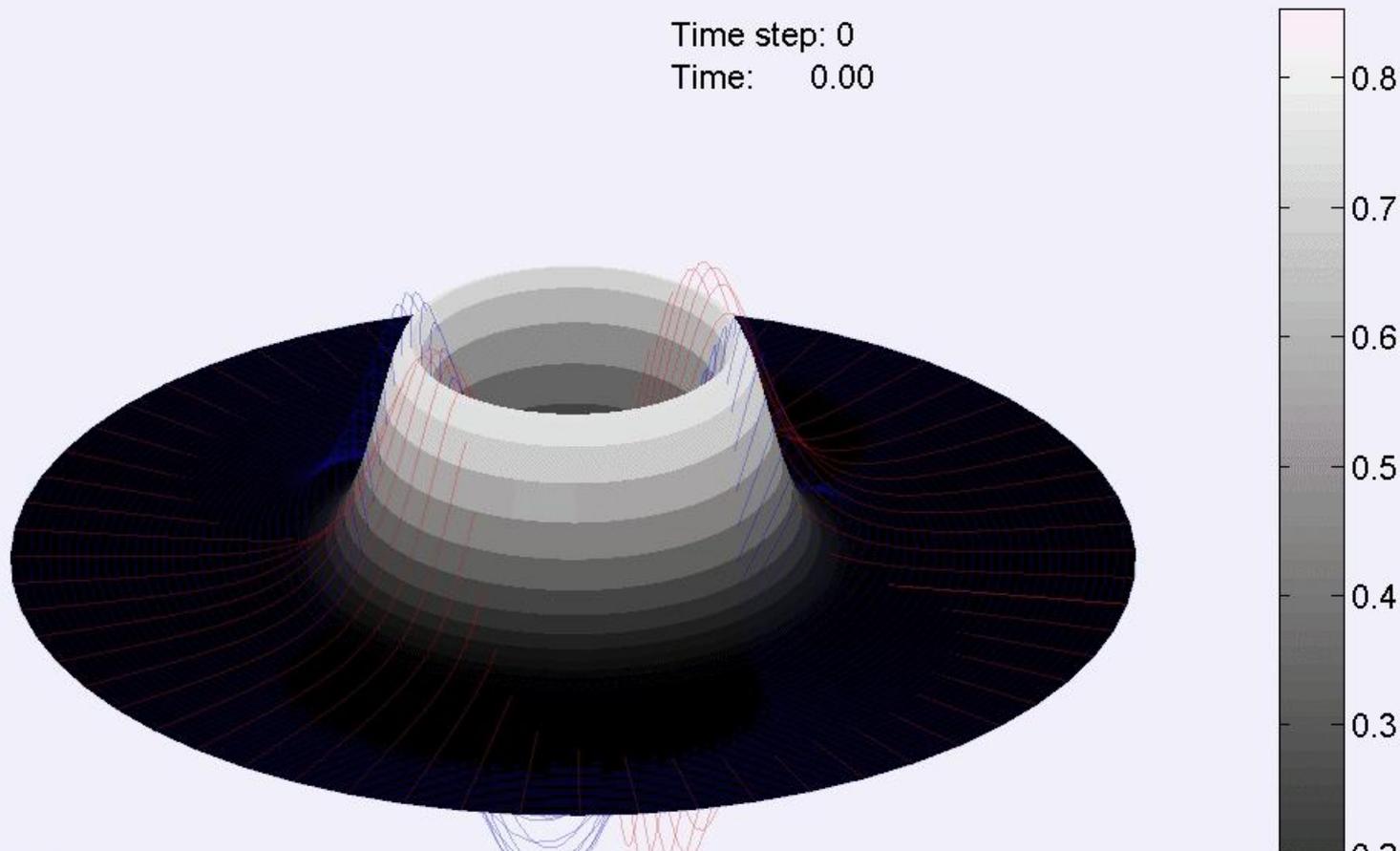
Introduction and Motivation

- Nonlinear Schrodinger Equation
 - Bose Einstein Condensates
 - Light propagation inside an optical fiber
 - Light propagation through a nonlinear crystal
- Vortex ring solutions
 - Different “charges”
 - Steady-state but typically unstable
- Nonlocality:
 - BEC: Thermal cloud – Uncondensed atoms in the condensate.
 - Crystals: Heat produced by laser changes properties of crystal.
- Purpose
 - To study azimuthal stability of vortex solutions, specifically to predict the critical mode where instability arises

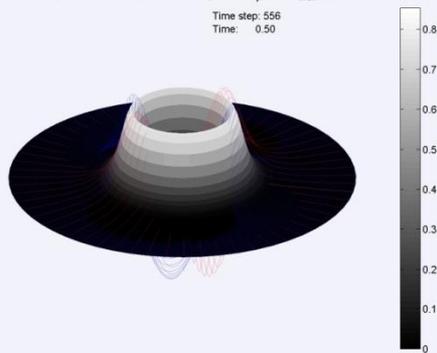
2D Local NLS in Polar Coords Order(4,2)

dt: 0.0009 B: 0.40825 m: 2 res_r: 60 res_p: 60 R_{max}: 20

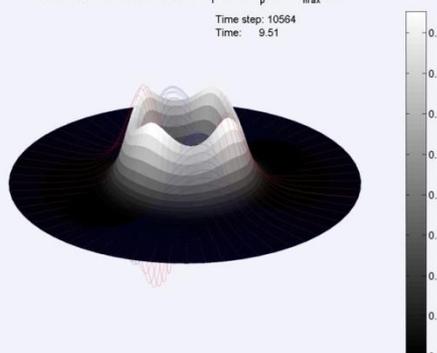
Time step: 0
Time: 0.00



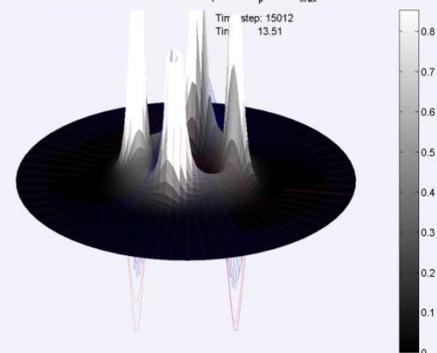
2D Local NLS in Polar Coords Order(4,2)
dt: 0.0009 B: 0.40825 m: 2 res_r: 60 res_p: 60 R_{max}: 20
Time step: 556
Time: 0.50



2D Local NLS in Polar Coords Order(4,2)
dt: 0.0009 B: 0.40825 m: 2 res_r: 60 res_p: 60 R_{max}: 20
Time step: 10564
Time: 9.51



2D Local NLS in Polar Coords Order(4,2)
dt: 0.0009 B: 0.40825 m: 2 res_r: 60 res_p: 60 R_{max}: 20
Time step: 15012
Time: 13.51



Local Theory

$$\text{NLS: } i\Psi_t + \nabla^2\Psi + s|\Psi|^2\Psi = 0$$

$$\text{Focusing Case: } s = +1$$

$$\text{Polar Laplacian: } \nabla^2\Psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\Psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\Psi}{\partial\theta^2}$$

Action Functional:

$$S = \int_0^\infty L dt$$

Lagrangian:

$$L = \int_0^{2\pi} \int_0^\infty \mathcal{L} r dr d\theta$$

Lagrangian Density:

$$\mathcal{L} = \frac{i}{2} (\Psi\Psi_t^* - \Psi^*\Psi_t) + \left| \Psi_r + \frac{1}{r}\Psi_\theta \right|^2 - \frac{s}{2} |\Psi|^4$$

Local Theory

Formulation of Quasi-1D Problem

Separable Solution: $\Psi(r, \theta, t) = f(r) A(\theta, t)$

$$f(r) \in \mathfrak{R}, \quad A(\theta, t) \in C$$

Plug into Lagrangian Density:

$$\mathcal{L} = |f(r)|^2 \frac{i}{2} (AA_t^* - A^* A_t) + \left| \frac{df}{dr} A + \frac{1}{r} f(r) A_\theta \right|^2 - |f(r)|^4 \frac{s}{2} |A|^4$$

Lagrangian with all r-dependence integrated out:

$$L = \int_0^{2\pi} \left(\frac{i}{2} C_1 (AA_t^* - A^* A_t) + C_2 |A|^2 + C_3 |A_\theta|^2 + C_5 A_\theta^* A + C_6 A_\theta A^* - \frac{s}{2} C_4 |A|^4 \right) d\theta$$

$$\begin{aligned} C_1 &= \int_0^\infty |f(r)|^2 r dr & C_2 &= \int_0^\infty \left| \frac{df}{dr} \right|^2 r dr \\ C_3 &= \int_0^\infty \frac{1}{r^2} |f(r)|^2 r dr & C_4 &= \int_0^\infty |f(r)|^4 r dr \\ C_5 &= \int_0^\infty \frac{1}{r} \frac{df}{dr} f(r)^* r dr & C_6 &= \int_0^\infty \frac{1}{r} \left(\frac{df}{dr} \right)^* f(r) r dr \end{aligned}$$

This is a new Lagrangian for a 1D PDE of A

Local Theory

Formulation of Quasi-1D Problem

A-equation is derived using functional derivative:

$$\frac{\delta S}{\delta A^*} = \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial [A_t^*]} + \frac{\partial}{\partial \theta} \frac{\partial \mathcal{L}}{\partial [A_\theta^*]} - \frac{\partial \mathcal{L}}{\partial A^*} = 0$$

$$i C_1 A_t = C_2 A - C_3 A_{\theta\theta} + (C_5 - C_6) A_\theta - s C_4 |A|^2 A$$

Note: $C_6 = C_5^*$

Since f is real, $(C_5 - C_6) = 0$

$$i C_1 A_t = C_2 A - C_3 A_{\theta\theta} - s C_4 |A|^2 A$$

Rescaling for simplicity: $A \rightarrow A e^{-i \frac{C_2}{C_1} t}$ $t \rightarrow \frac{C_3}{C_1} t$

Quasi-1D Problem
in Azimuthal Direction:

$$i A_t = -A_{\theta\theta} - s \frac{C_4}{C_3} |A|^2 A$$

Local Theory

Stability Analysis

$$iA_t = -A_{\theta\theta} - s \frac{C_4}{C_3} |A|^2 A$$

Plane-wave Solution (amp absorbed in f):

$$A(\theta, t) = e^{i(m\theta + \Omega t)}$$

Dispersion Relation:

$$\Omega = -m^2 + s \frac{C_4}{C_3}$$

Perturbed Plane Wave
 $|u|, |v| \ll 1$

$$A(\theta, t) = (1 + u(\theta, t) + iv(\theta, t)) e^{i(m\theta + \Omega t)}$$

Resulting PDE of u and v:

Set on Rotating frame:

$$t \rightarrow t + \frac{1}{2m} \theta$$

$$\begin{aligned} u_t &= -v_{\theta\theta} - \left[s \frac{C_4}{C_3} (2uv + u^2v + v^3) \right] \\ v_t &= u_{\theta\theta} + 2s \frac{C_4}{C_3} u + \left[s \frac{C_4}{C_3} (v^2 + 3u^2 + v^2u + u^3) \right] \end{aligned}$$

Expand u and v: $u(\theta, t) = \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} \hat{u}(K, t) e^{-iK\theta}$

$$v(\theta, t) = \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} \hat{v}(K, t) e^{-iK\theta}$$

Amplitudes of
Each mode:

$$\hat{u}(K, t) = \int_0^{2\pi} u(\theta, t) e^{iK\theta} d\theta$$

$$\hat{v}(K, t) = \int_0^{2\pi} v(\theta, t) e^{iK\theta} d\theta$$

Local Theory

Stability Analysis

Resulting PDE of Amplitudes:

$$\hat{u}_t = K^2 \hat{v} - \left[s \frac{C_4}{C_3} (2\hat{u} * \hat{v} + \hat{u} * \hat{u} * \hat{v} + \hat{v} * \hat{v} * \hat{v}) \right]$$

$$\hat{v}_t = \left(2s \frac{C_4}{C_3} - K^2 \right) \hat{u} + \left[s \frac{C_4}{C_3} (\hat{v} * \hat{v} + 3\hat{u} * \hat{u} + \hat{v} * \hat{v} * \hat{u} + \hat{u} * \hat{u} * \hat{u}) \right]$$

Convolution Terms: $\hat{a} * \hat{b}(K, t) = \sum_{K'=-\infty}^{\infty} \hat{a}(K', t) \hat{b}(K - K', t)$

Matrix form of Linearized PDE:

$$\begin{bmatrix} \hat{u}_t \\ \hat{v}_t \end{bmatrix} = \begin{bmatrix} 0 & K^2 \\ \left(2s \frac{C_4}{C_3} - K^2 \right) & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}$$

Remembering time-rescale, eigenvalues and normalized eigenvectors:

$$K_{\text{crit}} = \pm \sqrt{2s \frac{C_4}{C_3}}$$

$$\lambda_{1/2} = \pm \frac{C_3}{C_1} \sqrt{K^2 (K_{\text{crit}}^2 - K^2)}$$

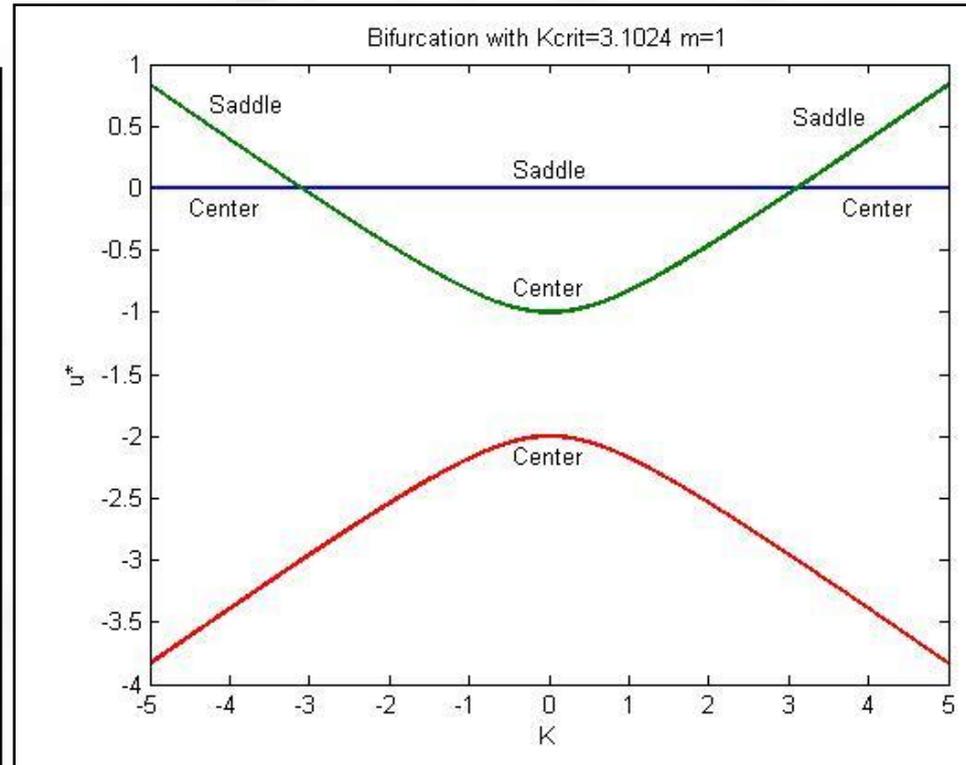
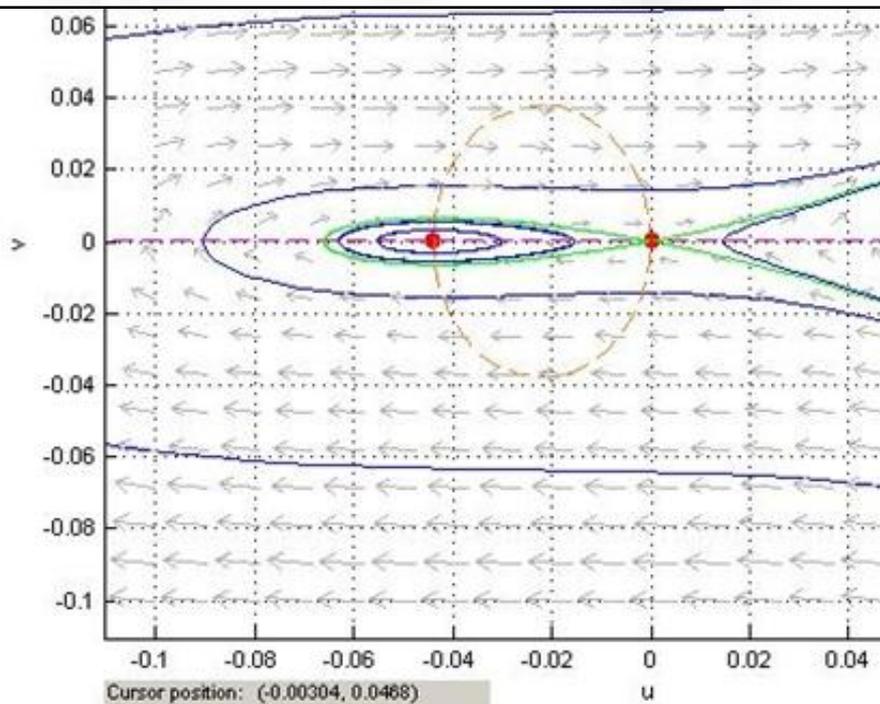
$$\mathbf{v}_{1/2} = \begin{bmatrix} \pm \frac{K}{K_{\text{crit}}} \\ \sqrt{1 - \left(\frac{K}{K_{\text{crit}}} \right)^2} \end{bmatrix}$$

Local Theory

Bifurcations and Nonlinear Effects

Ignoring inter-mode interaction, nonlinear PDEs of u and v yield the following bifurcations and phase-portrait (for a single mode):

$$\hat{v} = 0 \quad \hat{u} = -\frac{3}{2} \pm \frac{1}{2} \sqrt{1 + 8 \frac{K^2}{K_{Crit}^2}}$$



Therefore, near K_{crit} , theory may break down

Finding Steady-State Solutions

Variational Approach

Separable Steady-State: $\Psi(r, \theta, t) = f(r) e^{im\theta} e^{i\Omega t}$

Lagrangian Density is now: $\mathcal{L}(r, \theta) = \left(\Omega + \frac{m^2}{r^2} \right) |f(r)|^2 + \left| \frac{df}{dr} \right|^2 - \frac{s}{2} |f(r)|^4$

Ansatz: $f(r) = B r \exp\left(\frac{-r^2}{2\sigma^2}\right)$

Lagrangian is now:

$$L = \int_0^{2\pi} \int_0^\infty \mathcal{L}(r, \theta) r dr d\theta \longrightarrow L = \pi B^2 \sigma^2 \left[(m+1) - \left(\frac{s}{8} B^2 \sigma^2 - \Omega \right) \sigma^2 \right]$$

Time-independent Euler-Lagrangian Equations:

$$\frac{\partial L}{\partial B} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \sigma} = 0$$

Only nontrivial solution:

$$f(r) = B r \exp\left(\frac{-r^2}{2\sigma^2}\right) \quad \sigma = \frac{(8m+8)^{1/4}}{\sqrt{B}} \quad B = \frac{4\Omega}{\sqrt{2(1+m)}}$$

Finding Steady-State Solutions

Numerical Optimization Method

Separable Steady-State: $\Psi(r, \theta, t) = f(r) e^{im\theta} e^{i\Omega t}$

Insert into NLS, get ODE:

$$\vec{F}(f(r)) = - \left(\Omega + \frac{m^2}{r^2} \right) f(r) + \frac{1}{r} f'(r) + f''(r) + s |f(r)|^2 f(r) = 0$$

Gauss-Newton Nonlinear Equation Iterative Optimization Method:

Merit Function: $M(\vec{f}) = \frac{1}{2} \sum_{i=1}^n (F_i(\vec{f}))^2 \longrightarrow \nabla M(\vec{f}) = J(\vec{f})^T \vec{F}(\vec{f})$

Gauss-Newton Step:

$$\vec{p}_k = - (J_k^T J_k + \lambda_k I)^{-1} J_k^T \vec{f}_k$$

Forcing Term:

$$\lambda_k = 0.01$$

Backtracking Linesearch to get step size: $\min_{\alpha > 0} M(\vec{f}_k + \alpha \vec{p}_k) \rightarrow \alpha_k$

Init: $\vec{f}_0 = B r \exp \left(\frac{-r^2}{2\sigma^2} \right)$

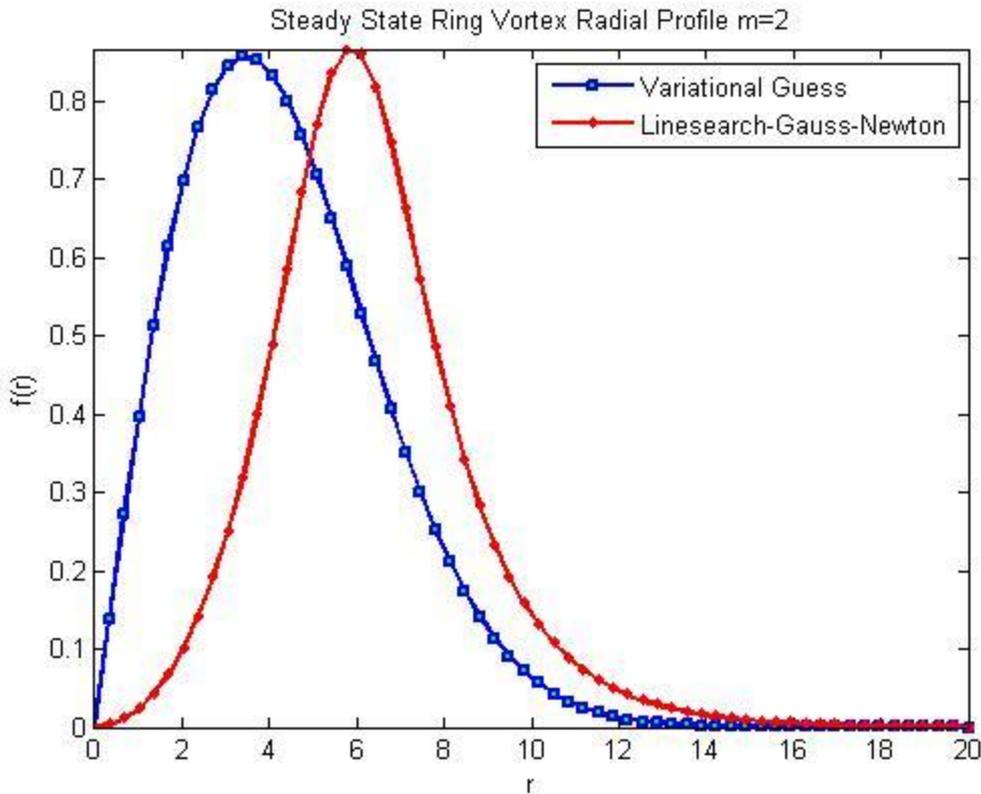
Step: $\vec{f}_{k+1} = \vec{f}_k + \alpha_k \vec{p}_k$

Stopping Criteria: $tol = \sqrt{\epsilon_{mach}}$

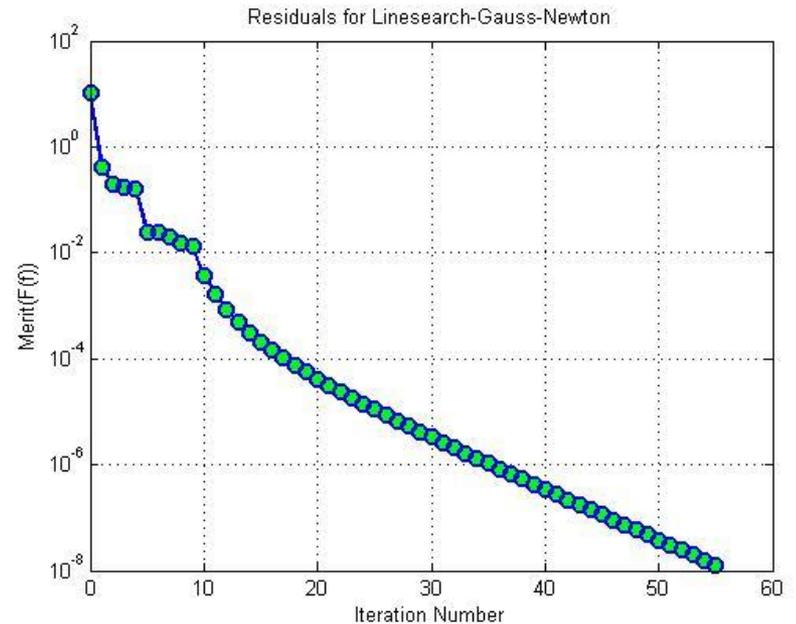
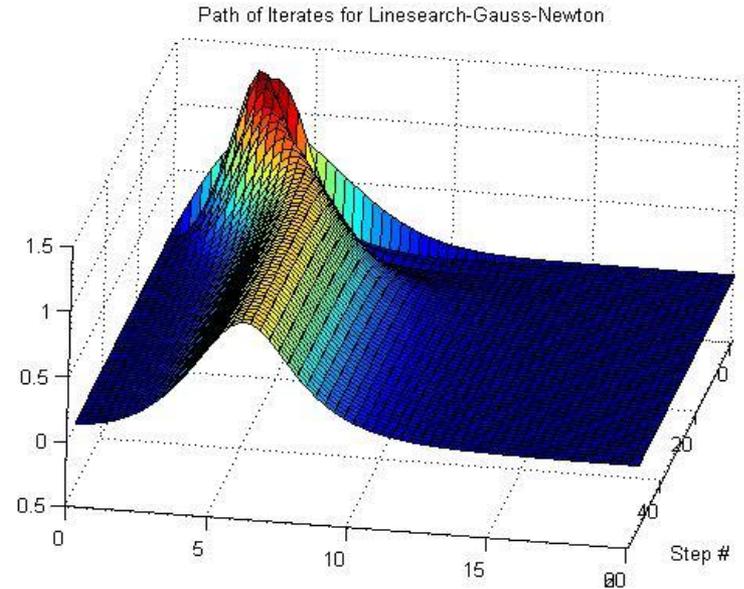
Finding Steady-State Solutions

Numerical Optimization Sample Result

$$m = 2 \quad \Omega = \frac{1}{4}$$



Total Steps: 55



Full System Simulation

Analytic Predictions

$$\text{VA Ansatz: } f(r) = B r \exp\left(\frac{-r^2}{2\sigma^2}\right)$$

$$C_3 = \int_0^\infty \frac{1}{r^2} |f(r)|^2 r dr = B (2m + 2)^{1/2}$$

$$C_4 = \int_0^\infty |f(r)|^4 r dr = B (2m + 2)^{3/2}$$

$$K_{\text{crit}} = 2\sqrt{m + 1}$$

VA is only close approximation for: $m = 1$

$$K_{\text{crit}} = 2\sqrt{2} = 2.8284$$

Numerical result: $K_{\text{crit}} = 3.1023$

VA gives good prediction for $m=1$, but not for higher charges.

With a better VA ansatz, K_{crit} could be predicted analytically for any charge.

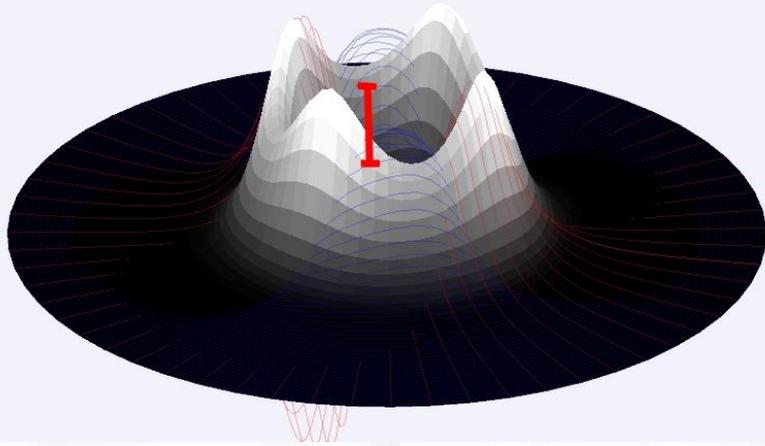
Full System Simulation

Numerical Method

PDE Integration:

4th order Runge-Kutta in time

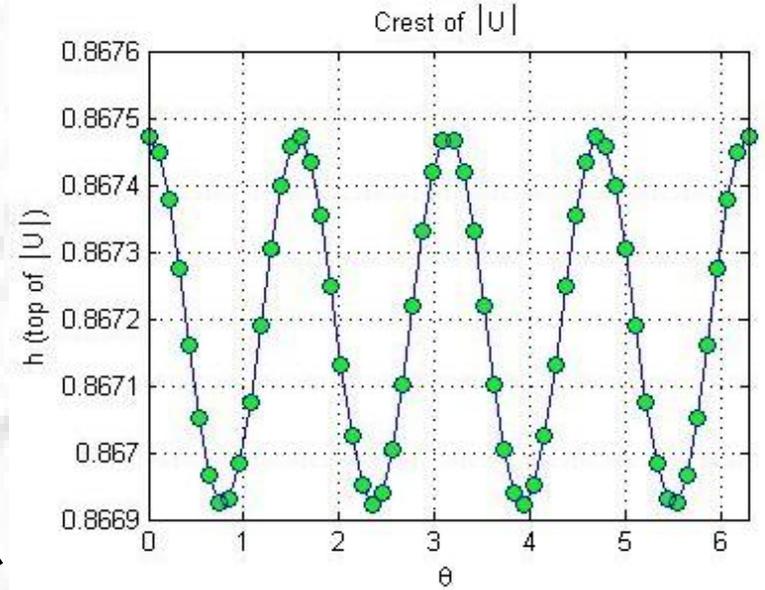
2nd order finite-difference on polar grid in space



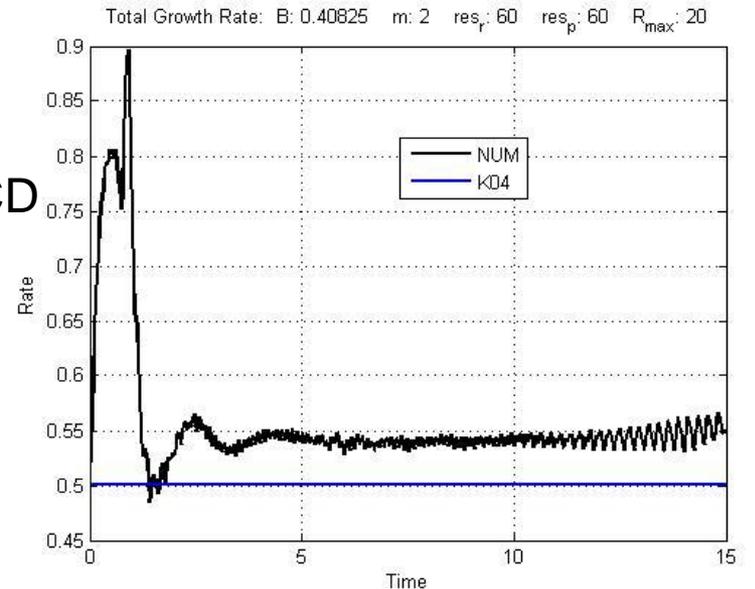
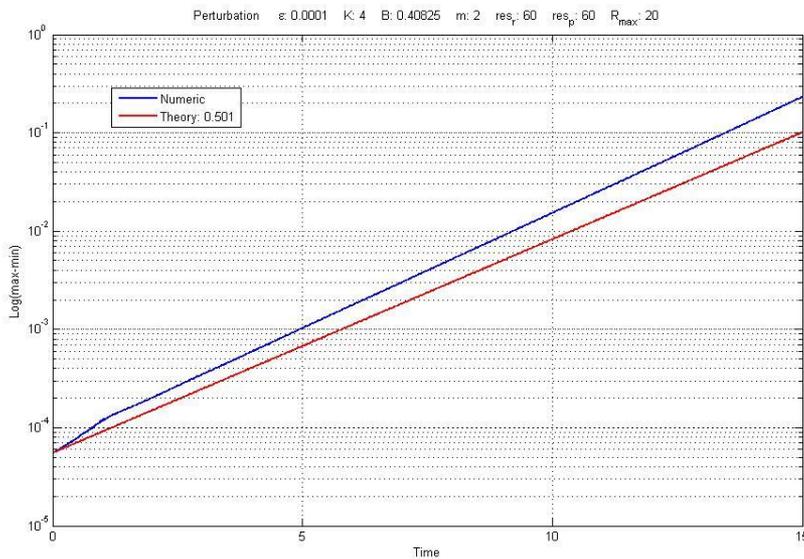
Extract
Crest:



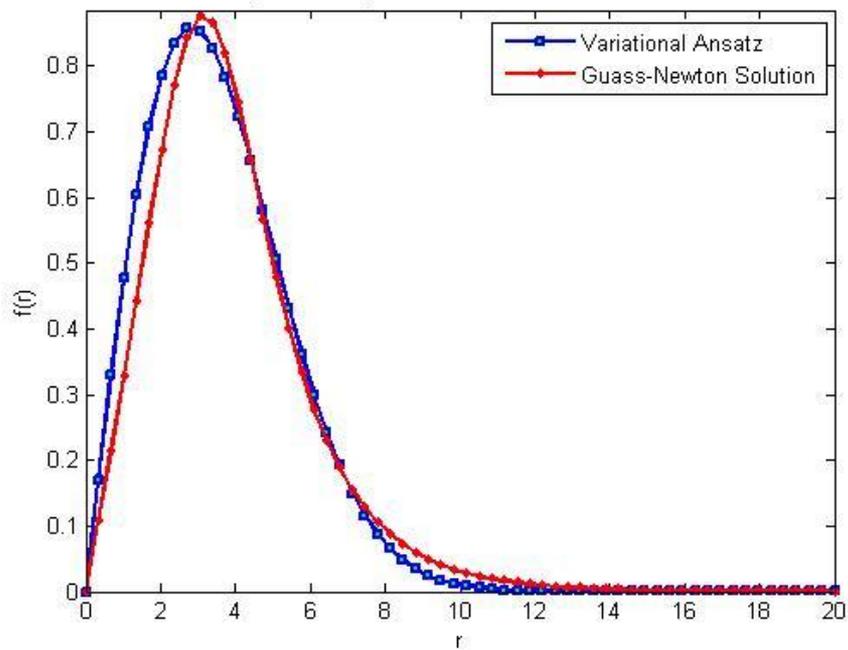
Record
Growth



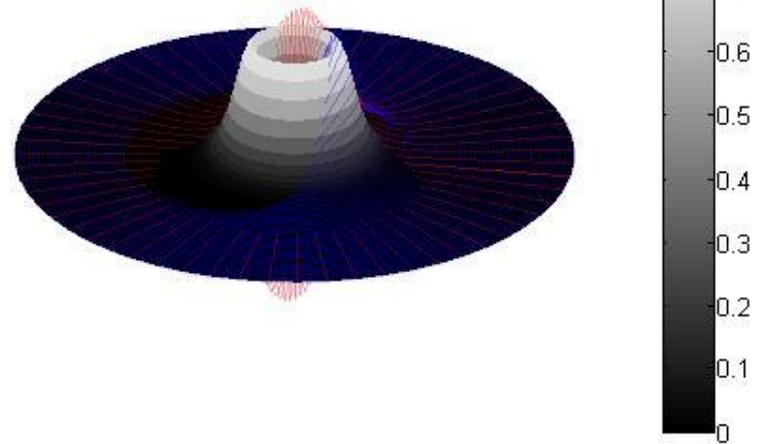
Calculate
Rate w/ CD



Steady State Ring Vortex Radial Profile $m=1$ $\Omega=0.25$



2D Local NLS in Polar Coords Order(4,2)
 dt: 0.0008 B: 0.5 m: 1 res_r: 60 res_p: 60 R_{max}: 20
 Time step: 0
 Time: 0.00



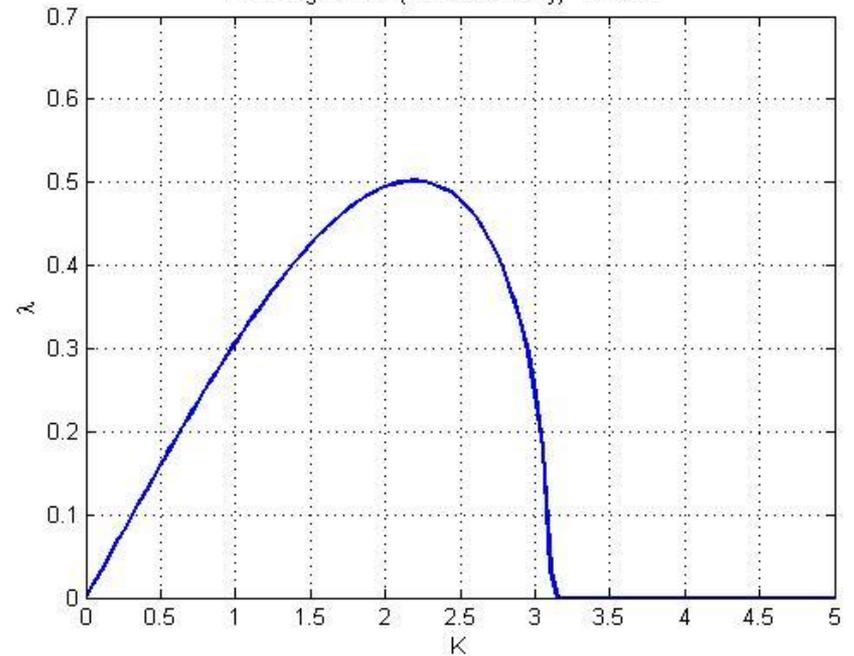
K

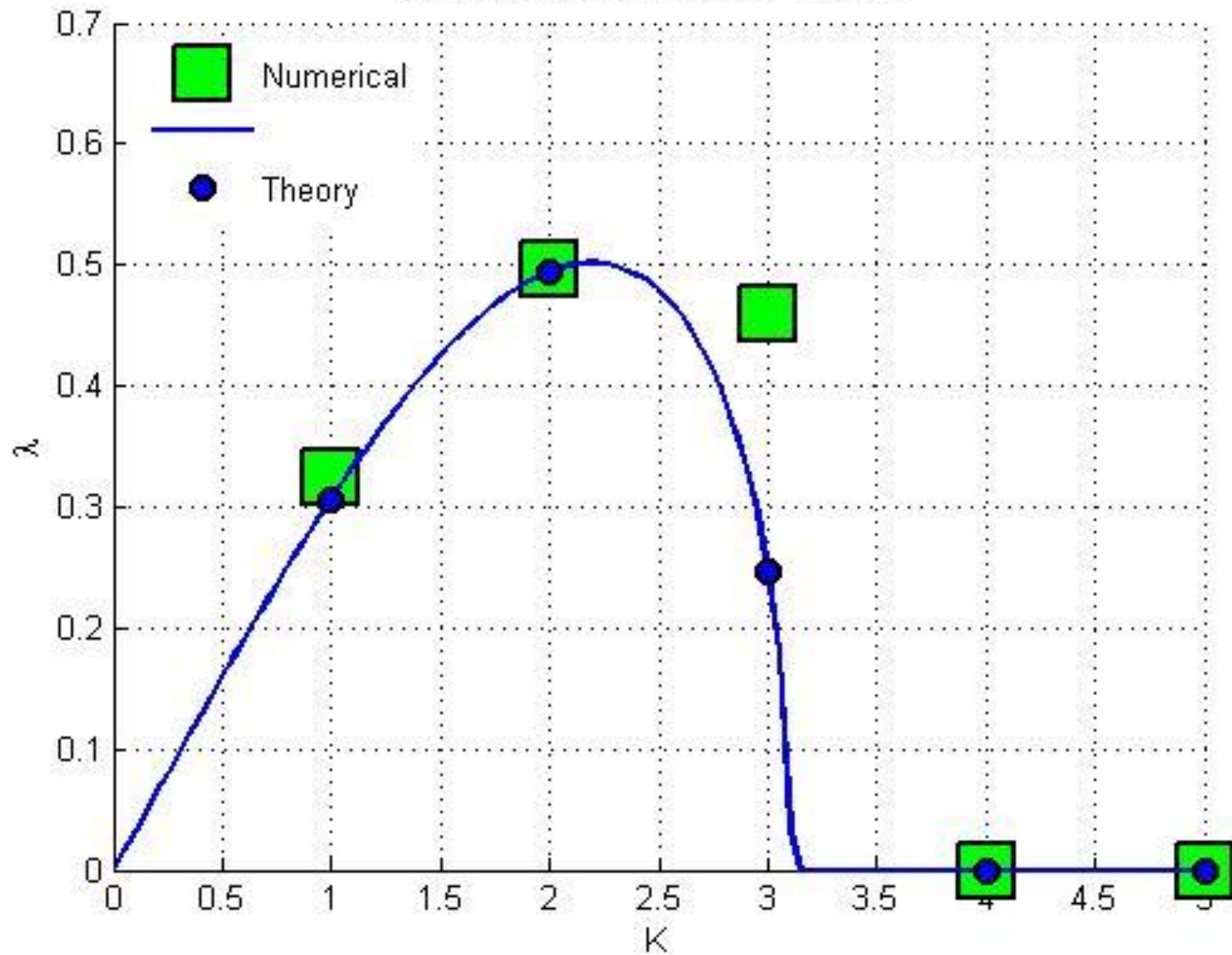
Eigenvalue

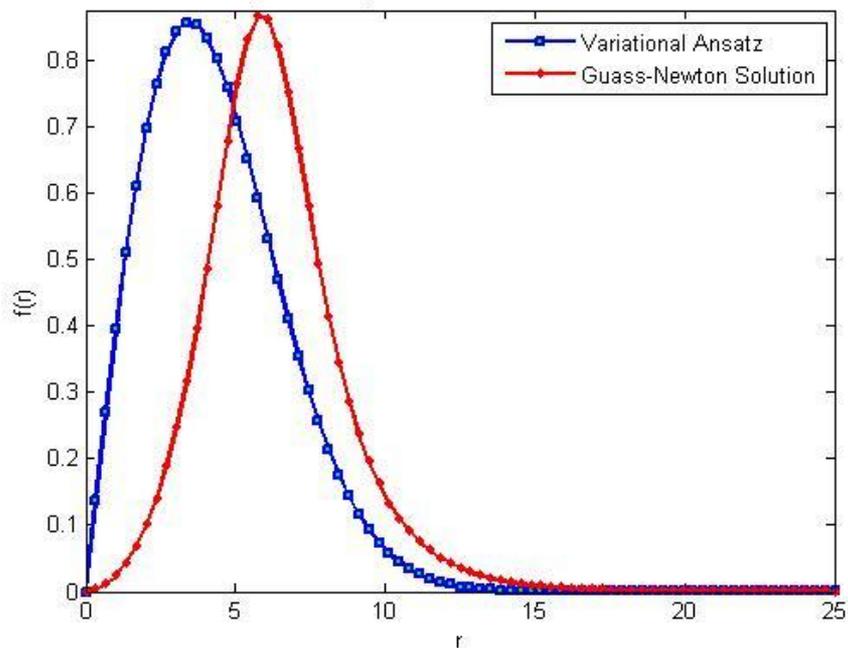
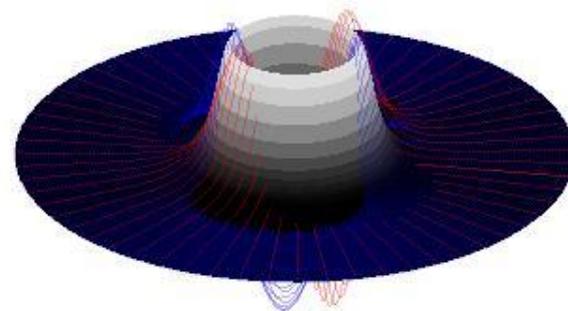
1	0.3063
2	0.4946
3	0.2465
4	1.0538 i
5	2.0453 i

Kcrit = 3.1017

K vs. Eigenvalue (Real $\lambda > 0$ only) B = 0.5



Growth Rate Results $m=1$ $\Omega=0.25$ 

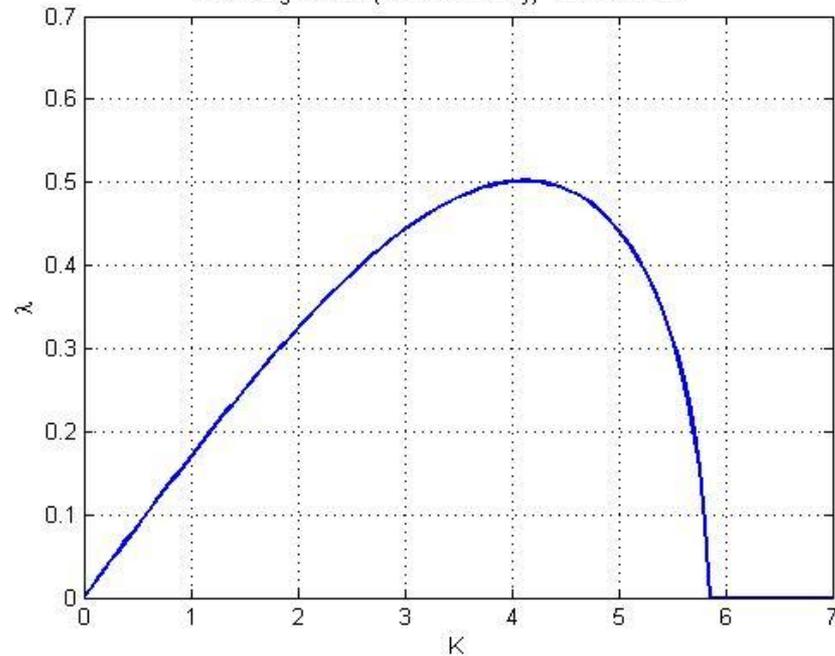
Steady State Ring Vortex Radial Profile $m=2$ $\Omega=0.25$ 2D Local NLS in Polar Coords Order(4,2)
dt: 0.0008 B: 0.40825 m: 2 res_r: 75 res_p: 60 R_{max}: 25Time step: 0
Time: 0.00

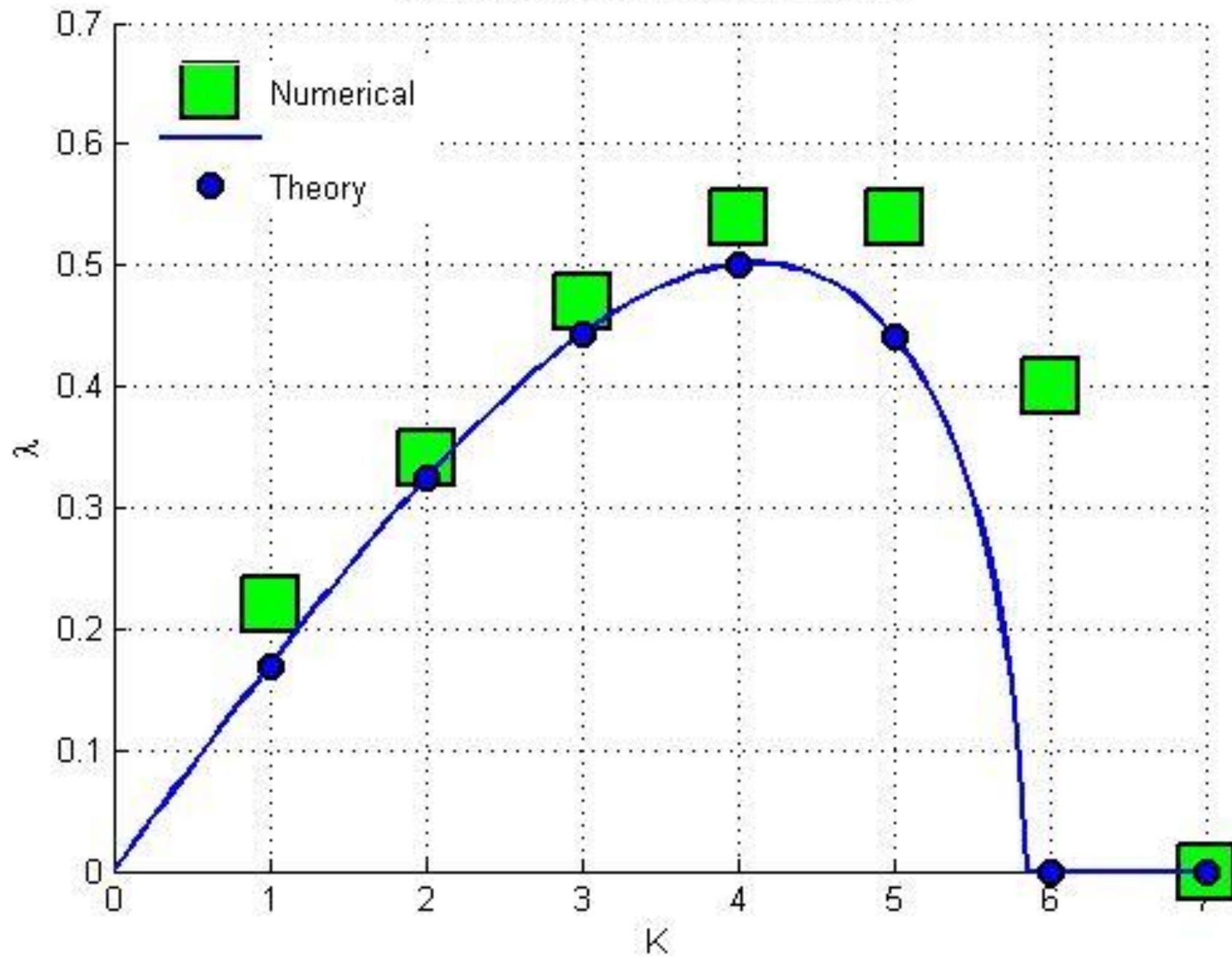
K

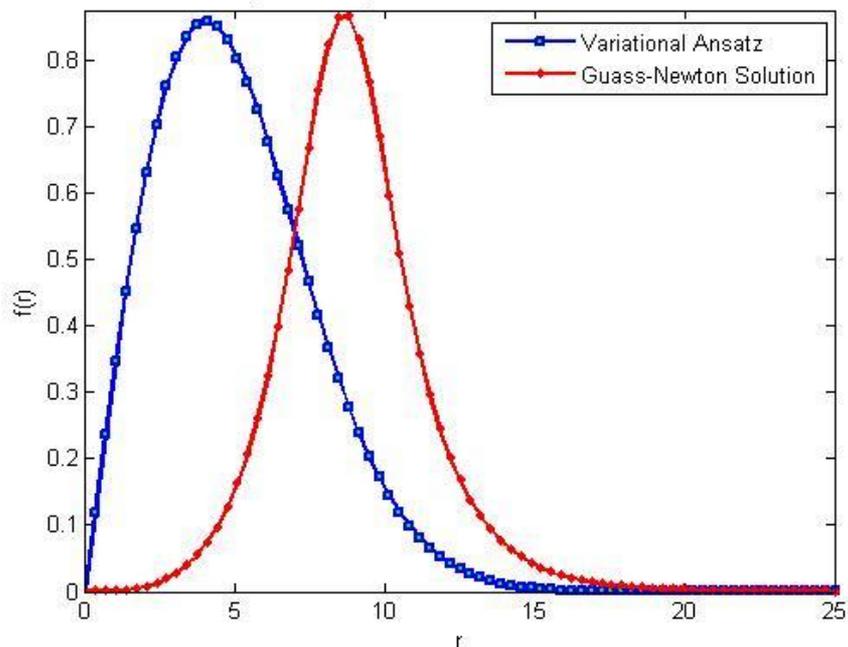
Eigenvalue

1	0.1698
2	0.3237
3	0.4431
4	0.5009
5	0.4416
6	0.2567 i
7	0.8047 i

Kcrit = 5.8231

K vs. Eigenvalue (Real $\lambda > 0$ only) B = 0.40825

Growth Rate Results $m=2$ $\Omega=0.25$ 

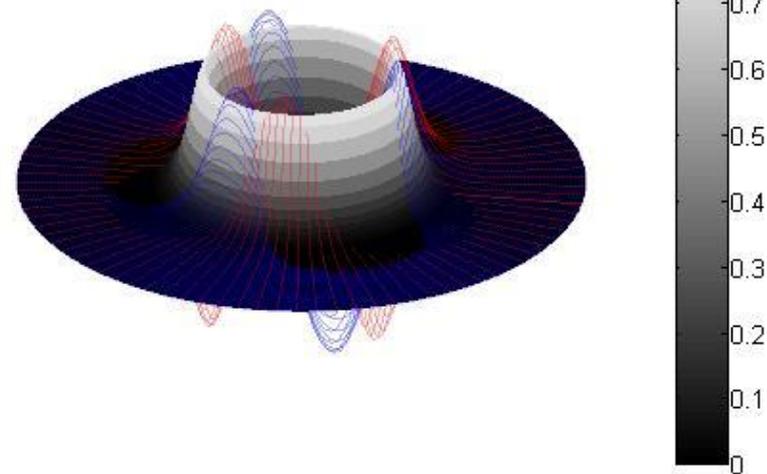
Steady State Ring Vortex Radial Profile $m=3$ $\Omega=0.25$ 

2D Local NLS in Polar Coords Order(4,2)

dt: 8e-005 B: 0.35355 m: 3 res_r: 75 res_p: 90 R_{max}: 25

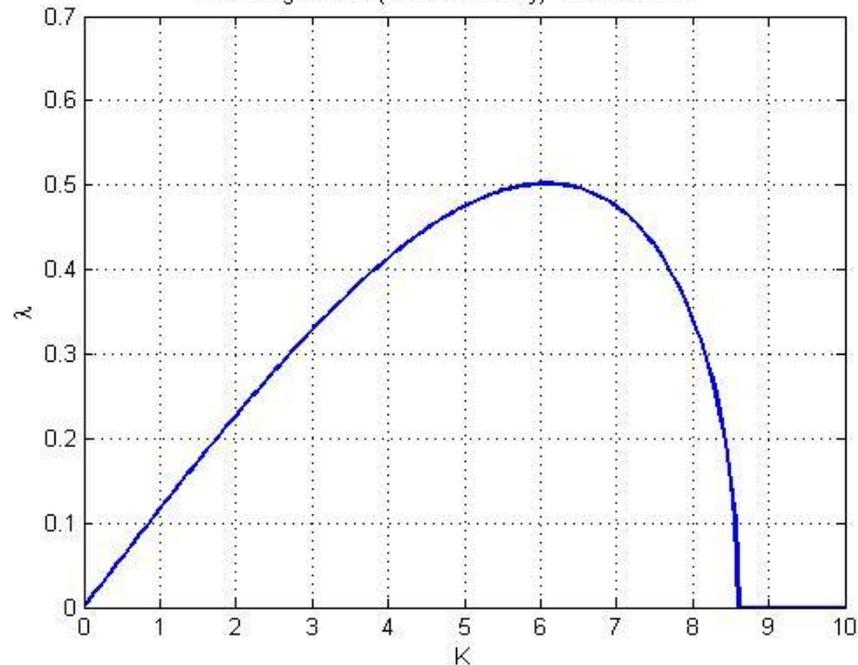
Time step: 0

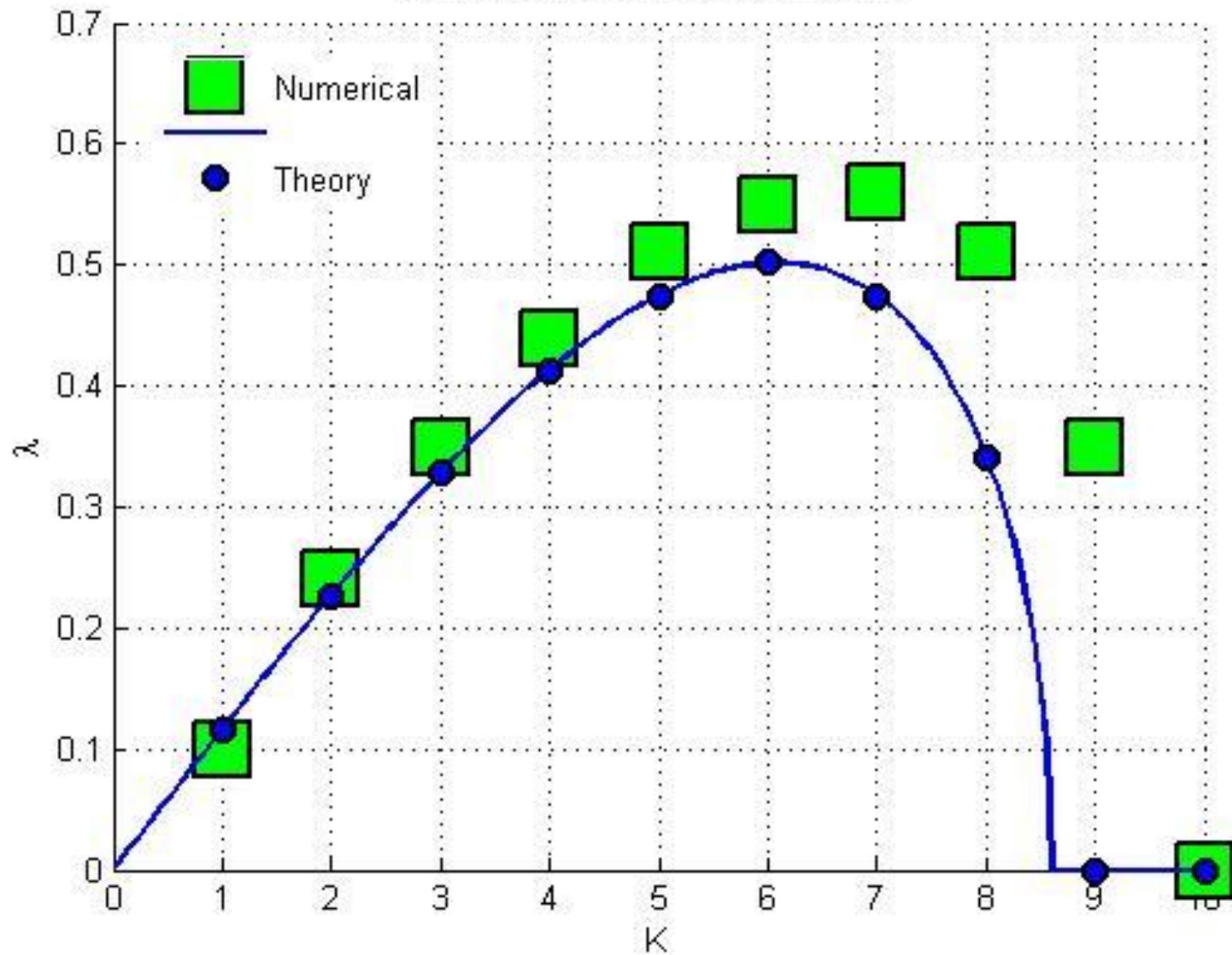
Time: 0.00



K	Eigenvalue
1	0.1160
2	0.2271
3	0.3282
4	0.4133
5	0.4748
6	0.5016
7	0.4743
8	0.3418
9	0.3258 i
10	0.6939 i

Kcrit = 8.5962

K vs. Eigenvalue (Real $\lambda > 0$ only) B = 0.35355

Growth Rate Results $m=3$ $\Omega=0.25$ 

Nonlocal Theory

$$i\Psi_t + \nabla^2\Psi + sN(|\Psi|^2)\Psi = 0$$

Nonlocal Nonlinearity:

$$N(|\Psi(r, \theta, t)|^2) = \int_0^{2\pi} \int_0^\infty V((r' - r), (\theta' - \theta)) |\Psi(r', \theta', t)|^2 r' dr' d\theta'$$

Nonlocal Response Function:
$$V(r, \theta) = \frac{1}{\pi\sigma^2} \exp\left(-\left[(r \cos \theta)^2 + (r \sin \theta)^2\right] \frac{1}{\sigma^2}\right)$$

Lagrangian Density:

$$\mathcal{L} = \frac{i}{2} (\Psi\Psi_t^* - \Psi^*\Psi_t) + \left| \Psi_r + \frac{1}{r}\Psi_\theta \right|^2 - \frac{s}{2} |\Psi|^2 N(|\Psi|^2)$$

Quasi-1D PDE:

$$i C_1 A_t = C_2 A - C_3 A_{\theta\theta} - s C(\theta, t) A$$

$$C(\theta, t) = \int_0^{2\pi} \int_0^\infty \int_0^\infty V((r' - r), (\theta' - \theta)) |f(r)|^2 |f(r')|^2 |A(\theta', t)|^2 r r' dr dr' d\theta'$$

Nonlocal Theory

Define:

$$R(\theta' - \theta) = \int_0^\infty \int_0^\infty V((r' - r), (\theta' - \theta)) |f(r)|^2 |f(r')|^2 r r' dr dr'$$

C is now a convolution term:

$$C(\theta, t) = \int_0^{2\pi} R(\theta' - \theta) |A(\theta', t)|^2 d\theta' = R * |A|^2$$

New PDE:
$$iA_t = -A_{\theta\theta} - \frac{s}{C_3} A(R * |A|^2)$$

In stability procedure,
add in this transform:
$$\hat{R}(K) = \int_0^{2\pi} R(\theta) e^{iK\theta}$$

Eigenvalues and Critical Mode:

$$\lambda_{1/2} = \pm \frac{C_3}{C_1} \sqrt{K^2 \left(2s \frac{\hat{R}(K)}{C_3} - K^2 \right)}$$

$$K_{\text{crit}} = \pm \sqrt{2s \frac{\hat{R}(K)}{C_3}}$$

Thus, instability can theoretically be damped and even eliminated

Conclusion and Further Research

For local case, results are very close considering the bifurcation at K_{crit}

From nonlocal theory, it seems that K_{crit} can be altered, enhancing stability depending on the nonlocal response function

Further numeric work needs to be done to test nonlocal theory

