Numerical Computation of Geodesics on Combined Piecewise Smooth Surfaces

Ron Caplan, Michael Davis, Jenny Dehner, Nick Hooper, Heather Mattie

Advisor: Ali Nadim   Boeing Liaison: Tom Grandine

Claremont Graduate University

Boeing Clinic 08-09

Claremont Graduate University

- 2188 graduate students

- A member of the Claremont Colleges
  - **Pomona College**
  - **Harvey Mudd College**
  - **Claremont McKenna College**
  - **Etc.**
- School of Mathematical Sciences
  - **Mathematics Clinic**

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The Proposed Problem

http://en.wikipedia.org/wiki/File:Boeing_787.jpg

- Boeing 787
  - **composite tape**
  - **must be laid flat**
  - **no wrinkles, creases**

- Laying each strip along a geodesic minimizes creasing and wrinkling

- Need an algorithm that can aid in the process of finding geodesics along the combined surfaces that make up a fuselage

Overview

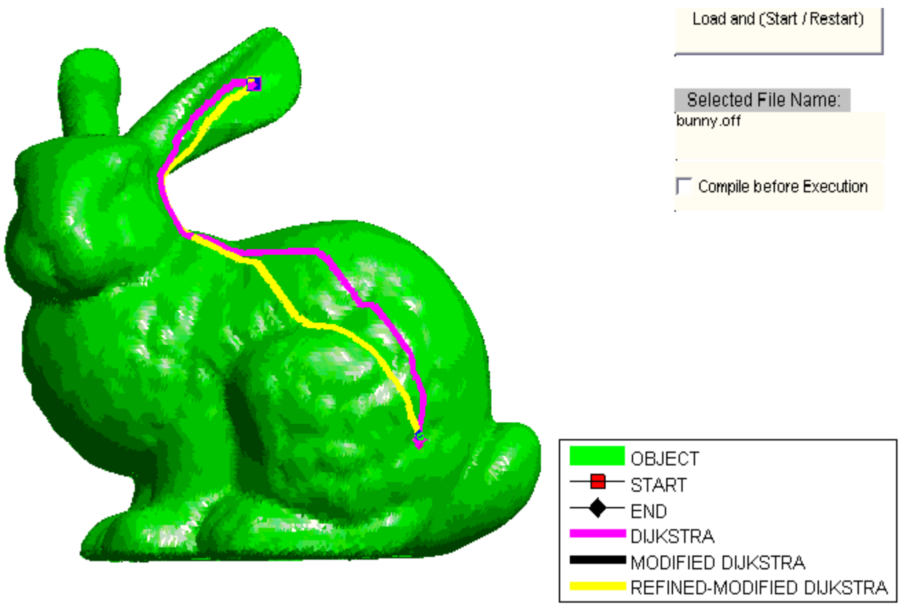
- Foundational work
- The main algorithm
- A few test runs
- Challenges and improvements
- Conclusions

Getting Acquainted with Geodesics

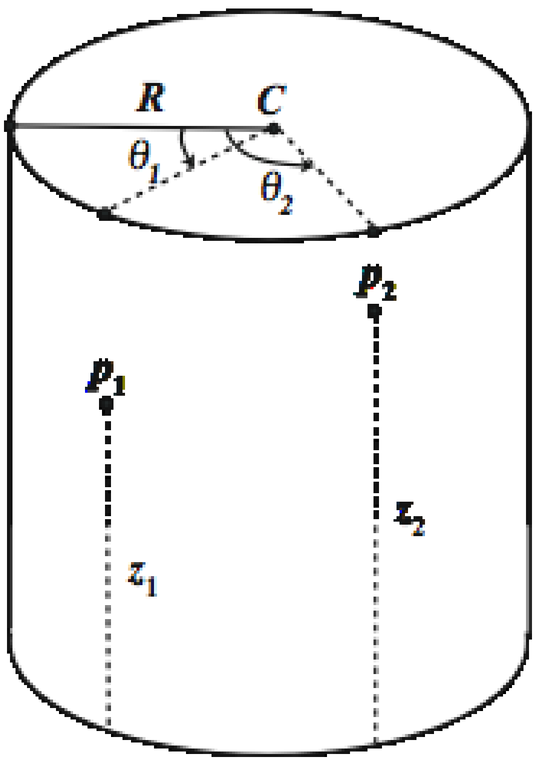
- How we got started:
  - Reviewed previous work from CGU Boeing Clinic Team 07-08
  - Performed literature review
  - Reviewed topics in differential geometry
  - “Evolution” algorithm seemed best path to follow

Previous Team’s Work

- Dijkstra’s Algorithm
  - **Limited to points and edges on the mesh**
- Fast Marching Method and Level Set Method
  - **Difficult to program and run**
- Modified Dijkstra’s Algorithm
  - **Does not take the curvature of the surface into account**
  - **Can only be used on meshed surfaces**



Toy Problem: A Cylinder



- Our first simple introduction to shortest paths and geodesics
  - **Compute shortest paths analytically using distance formulas**
  - **Gives us a reference for methods developed later**

Our New Method

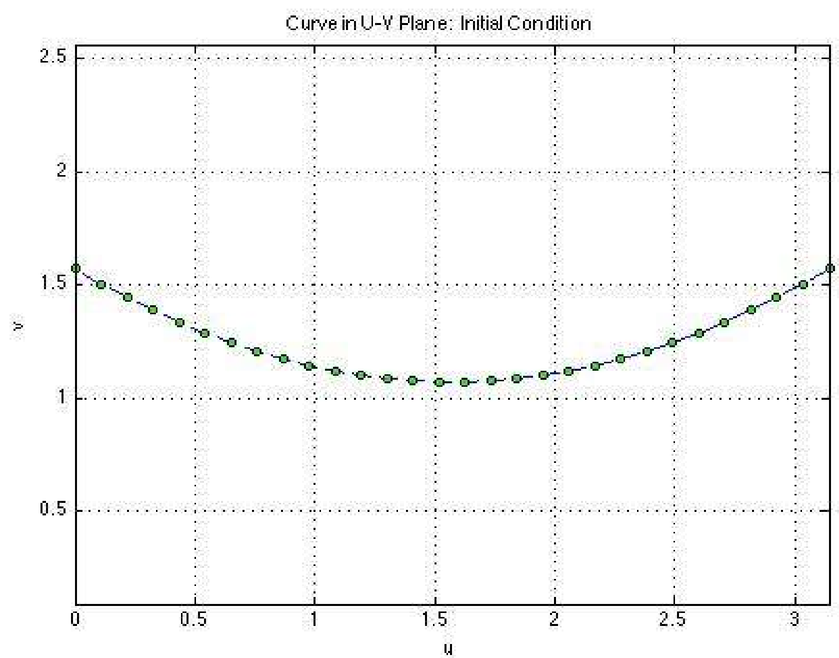
- An algorithm for evolution of an initial curve toward the geodesic
- Achieved by computing numerical properties of the curve and moving marker points along the curve in proportion to the geodesic curvature vector

The Evolution Algorithm

- Given some 3-D surface parameterized by [redacted] represented by the equation:

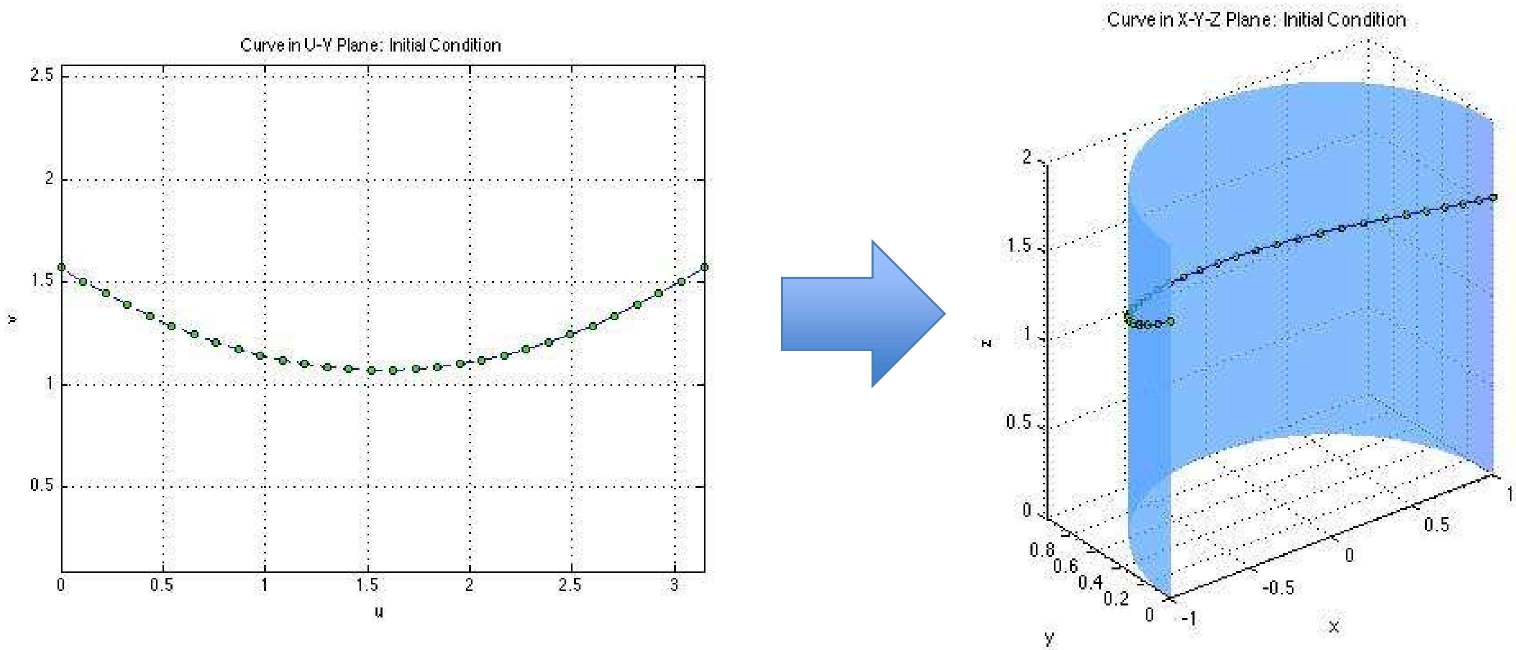
[redacted]

the initial point [redacted] and the end point [redacted], connect the two points in 2-dimensional space with [redacted] , some smooth initial curve. (Many times, a straight line between the points is the easiest.)



The Evolution Algorithm

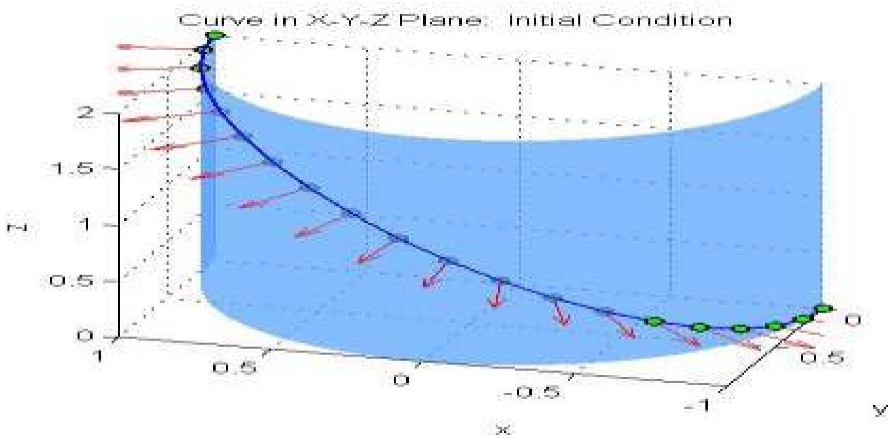
- Discretize the smooth curve γ, parameterized by arclength, into N points such that $\gamma(s_i)$ refers to the i^{th} point in the list of points and map these points onto the surface.



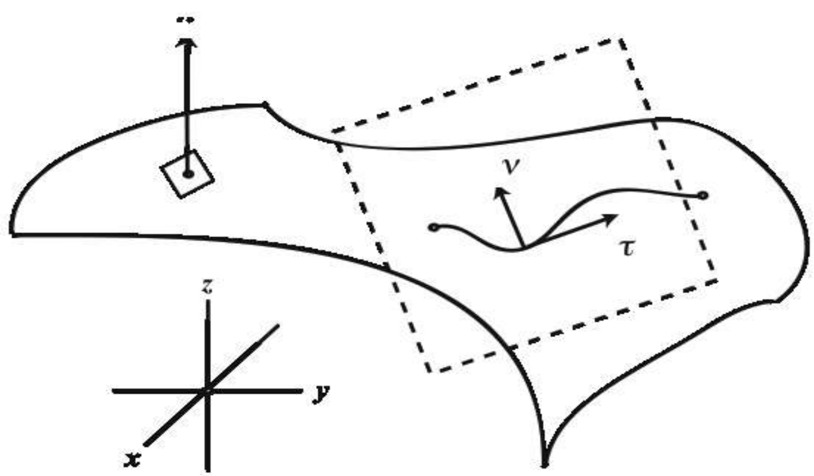
- Calculate arclength at each point in the u, v plane and on the 3-D surface.

The Evolution Algorithm

- In 3-dimensional space, calculate the normal $\hat{\mathbf{n}}$ to the surface at every point on the curve.



$$\hat{\mathbf{n}} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left|\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|\right|}$$



ɔ in 3-dimensional space, calculate the /ature vector:

$$\mathbf{k} = \frac{\partial \tau}{\partial s} = \frac{\partial^2 x}{\partial s^2}\hat{\mathbf{i}} + \frac{\partial^2 y}{\partial s^2}\hat{\mathbf{j}} + \frac{\partial^2 z}{\partial s^2}\hat{\mathbf{k}}$$

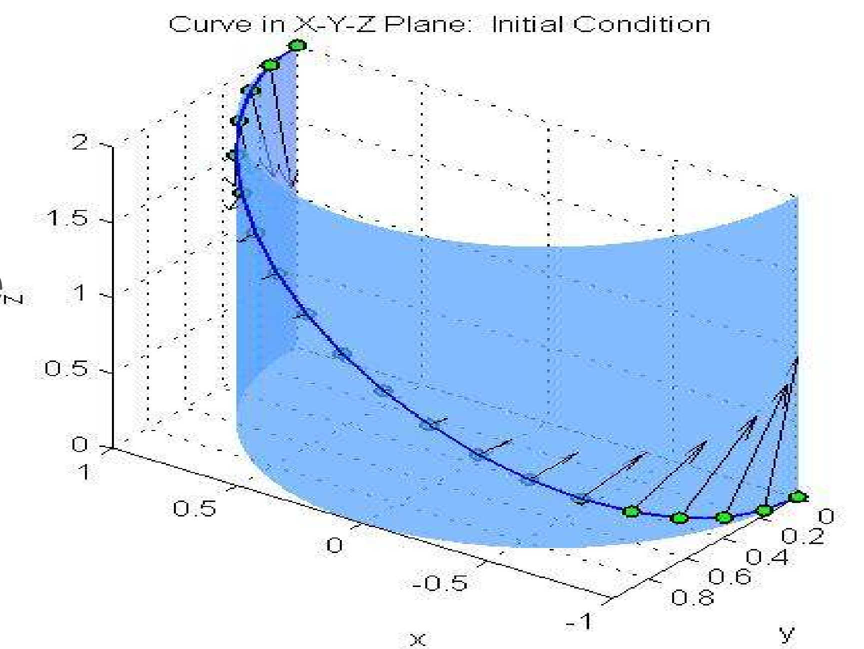
The Evolution Algorithm

- From the previous steps, calculate the *geodesic curvature* [redacted] at every point.

[redacted]

- Geodesic occurs when: [redacted]
- *Stopping Criterion*:
  - **Check sum of squares of 2-norm. If it is larger than some tolerance, take a evolution step.**

$$\left(\sum_{i=1}^{n} |\mathbf{k_g}(\mathbf{s_i})|^2\right)^{1/2} = \epsilon_{\text{tol}} \approx 0$$



The Evolution Algorithm

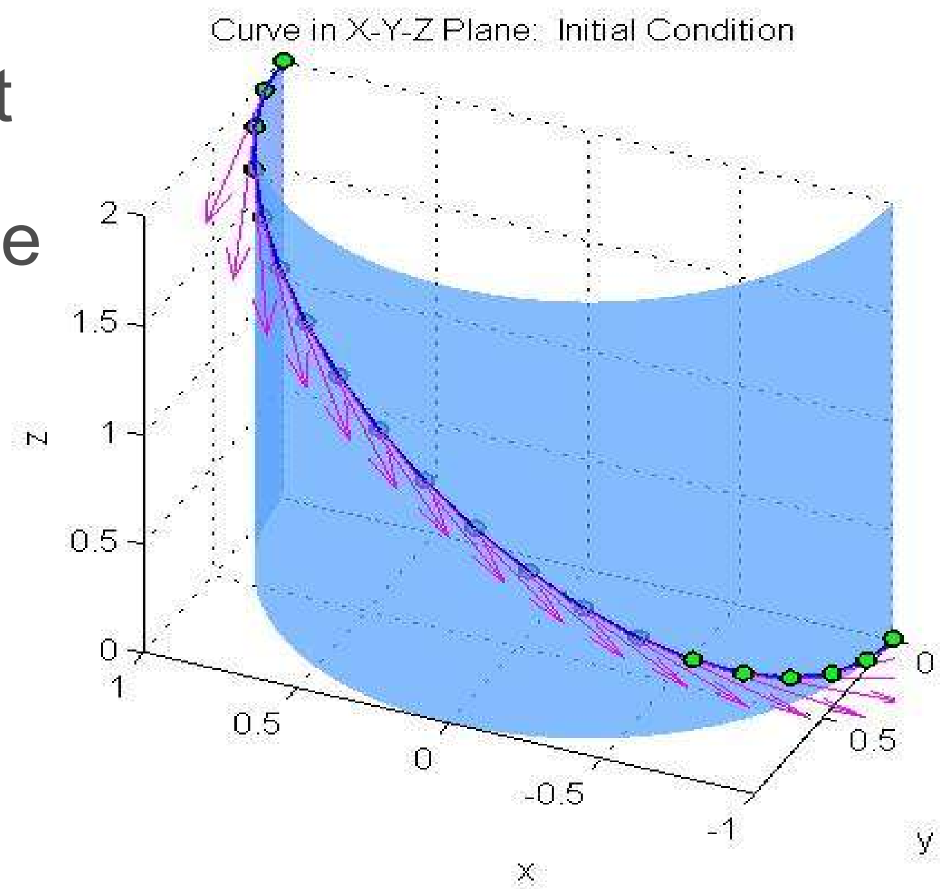
- Compute the tangent vector to the curve at each point

$$\tau = \frac{\partial x}{\partial s}\hat{\mathbf{i}} + \frac{\partial y}{\partial s}\hat{\mathbf{j}} + \frac{\partial z}{\partial s}\hat{\mathbf{k}}$$

- Calculate a signed scalar quantity at each point that represents a velocity with which each point will move

$$M_s = (\hat{\mathbf{n}} \times \hat{\tau}) \cdot \mathbf{k_g}$$

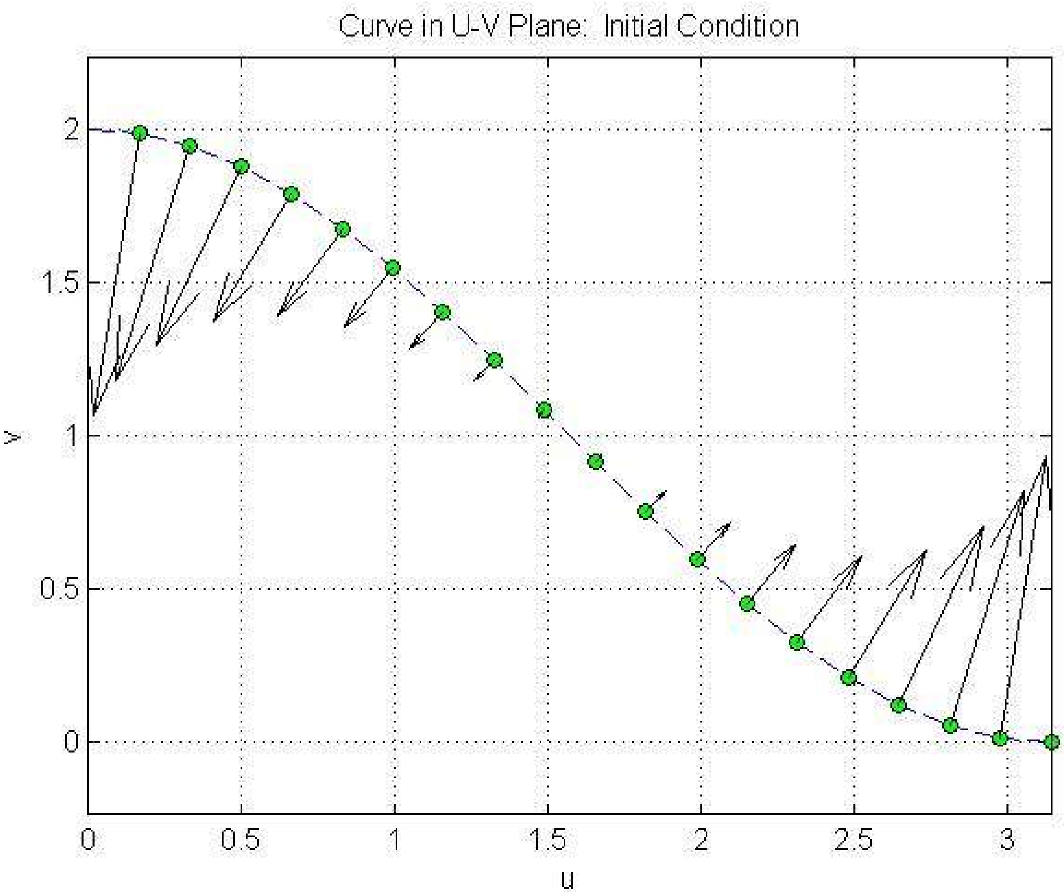
- At every point along the curve in the u, v plane calculate a local normal to the curve $\hat{\mu}$



The Evolution Algorithm

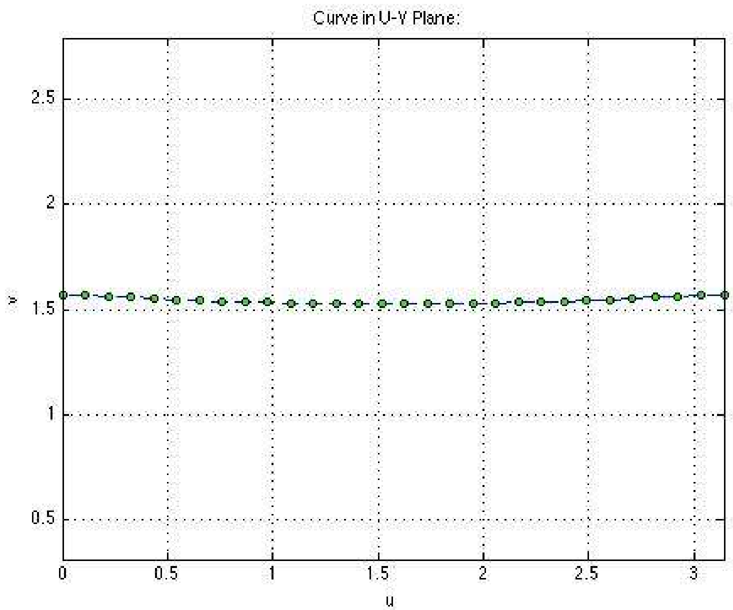
- Move each point in the [redacted] plane in the direction of the normal vector to the curve and proportional to the calculated velocity

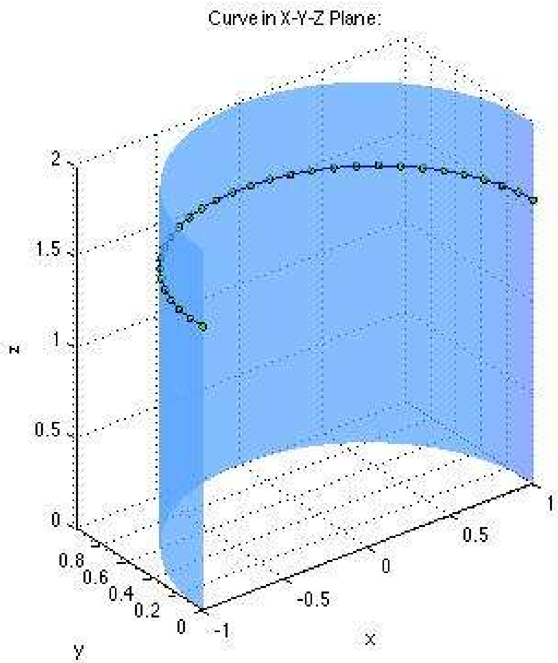
$$\frac{d\mathbf{r}}{dt} = \hat{\mu}\mathbf{M_s}$$



The Evolution Algorithm

- Repeat previous steps until the stopping criterion ($\left(\sum_{i=1}^{n} |\mathbf{k_g(s_i)}|^{\mathbf{2}}\right)^{1/2} = \epsilon_{tol} \approx 0$) is satisfied:





Implementation

- Derivatives Subroutine (CGUBC_deriv)
  - **Central Difference Method**
    - Calculates first derivatives quickly

$$\mathbf{f'(x)} \approx \frac{\mathbf{\Delta x_L}}{\mathbf{\Delta x_L + \Delta x_R}}\left(\frac{\mathbf{\Delta f_R}}{\mathbf{\Delta x_R}}\right) + \frac{\mathbf{\Delta x_R}}{\mathbf{\Delta x_L + \Delta x_R}}\left(\frac{\mathbf{\Delta f_L}}{\mathbf{\Delta x_L}}\right)$$

    - Computes second derivatives with low accuracy
  - **Cubic Spline Method**
    - Numerically calculates more accurate second derivatives
    - Increases complexity of subroutine

Implementation

- Time Integration:
- 4th Order Runga-Kutta:

$$\mathbf{r^{n+1}} = \mathbf{r^{n}} + \frac{\boldsymbol{\Delta}\mathbf{t}}{\mathbf{6}}(\mathbf{k_1} + \mathbf{2k_2} + \mathbf{2k_3} + \mathbf{k_4}),$$

$$k_1 = F(\mathbf{r^n}),$$

$$k_2 = F(\mathbf{r^n} + \frac{\boldsymbol{\Delta}\mathbf{t}}{\mathbf{2}}\mathbf{k_1}),$$

$$k_3 = F(\mathbf{r^n} + \frac{\boldsymbol{\Delta}\mathbf{t}}{\mathbf{2}}\mathbf{k_2}),$$

$$k_4 = F(\mathbf{r^n} + \boldsymbol{\Delta}\mathbf{t}\mathbf{k_3}),$$

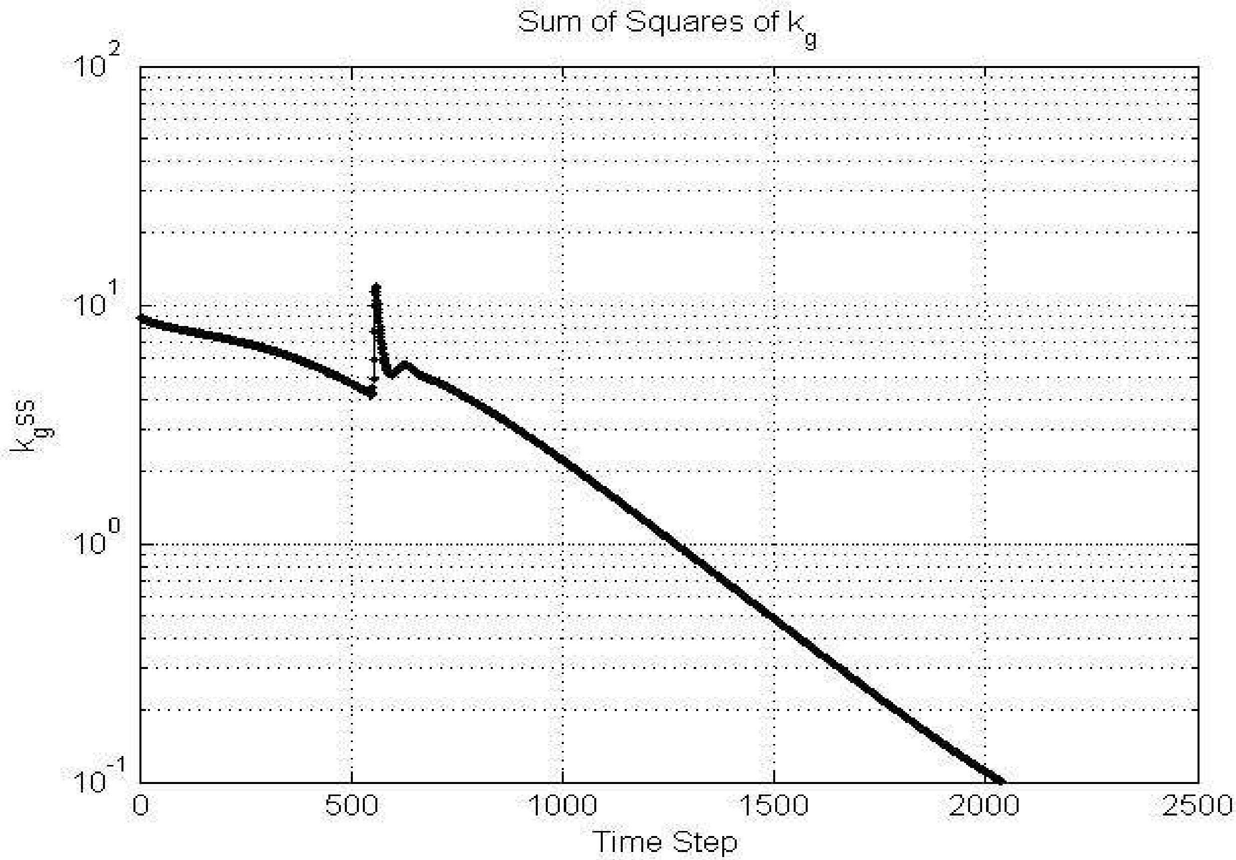
- Time step needs to be chosen carefully

Example #1

- Surface: Cylinder
- Initial condition near geodesic

Example #2

- Surface: Cylinder
- Complicated initial condition
- We see clumping of points and instability “blip”



Example #3a

- Surface: Cylinder
- We see boundary condition enforcement
- Clumping, but no blip.

Example #3b

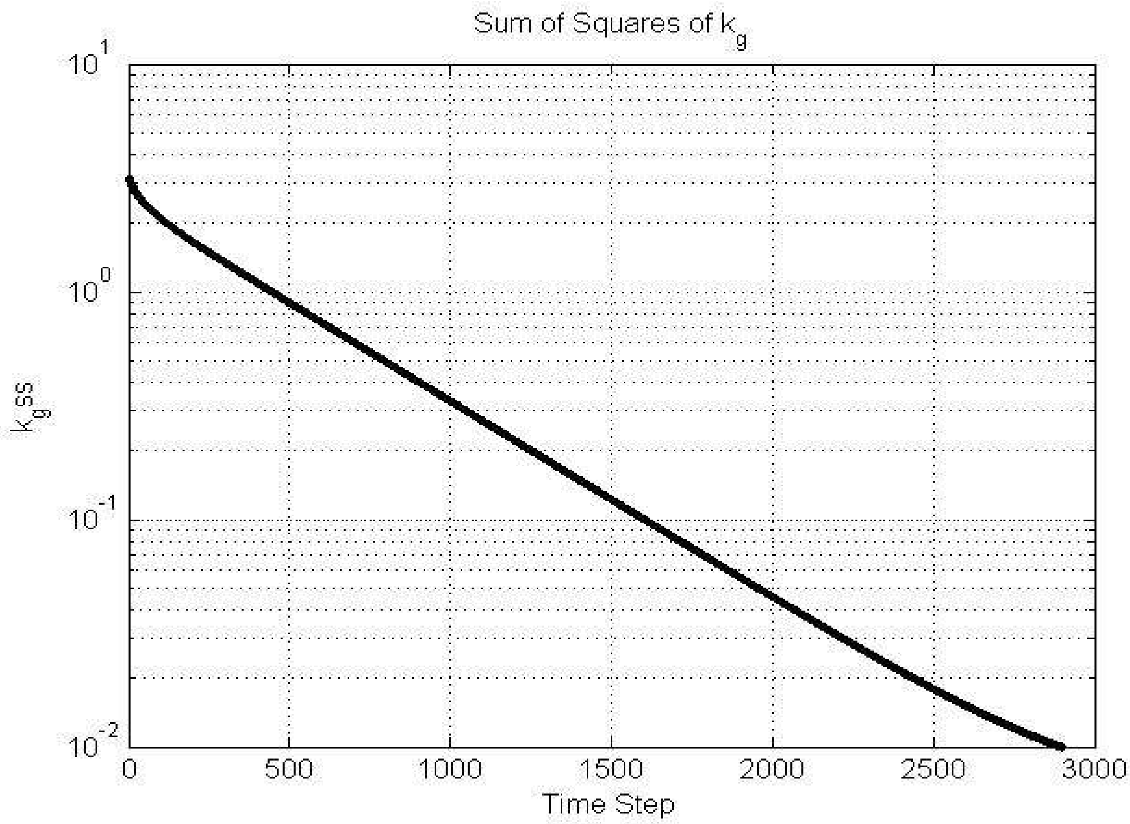
- Surface: Cylinder
- Using cubic spline derivatives
  - **Less clumping**

Example #3c

- Surface: Cylinder
- Using u-redistribution
  - **No clumping**

Example #4

- Surface: Sphere
- More sensitive
- Still works well for “nice” initial curves away from pole



Challenges & Solutions

- Challenges
  - **Convergence rate slow for some problems**
  - **Stability requirements on time step**
  - **“Singular” points cause breakdown**

- Possible solutions
  - **Implicit time scheme**
  - **Manual/automatic singular point computations**

Conclusions / Results

- Current evolution process can be used to relax an initial curve toward a geodesic
- Some issues remain to be addressed
- In principle, the algorithm should be able to handle constraints (with properly modified stopping criterion) and multiple surfaces

Future Work

- Constraints on surfaces
  - **Include regions that are inaccessible**
- Multiple surfaces
  - **Crossing edges**
- Other test cases
  - **Objects with corner**

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