# Chaotic Stellar Dynamo Models

Math 638 Final Project

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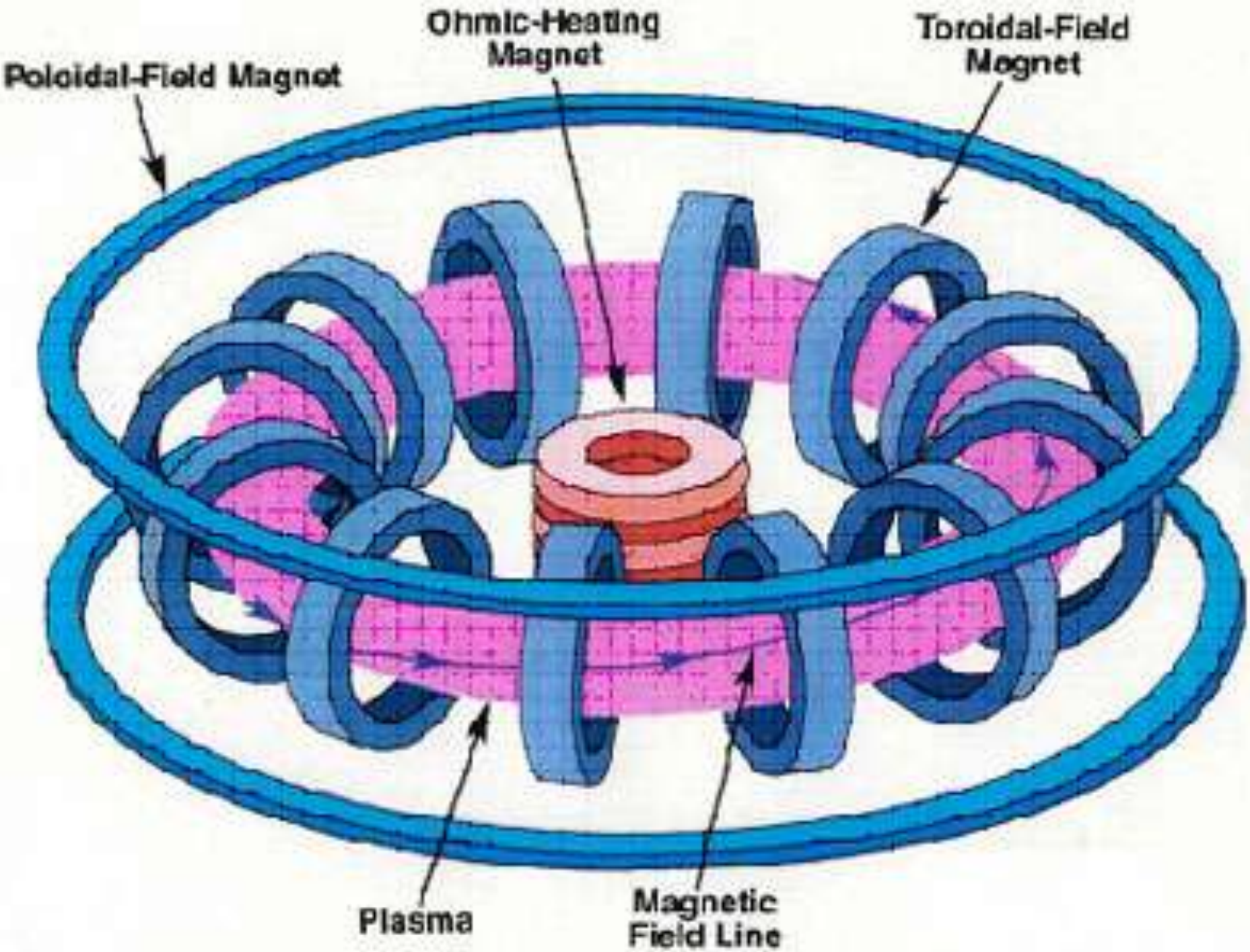
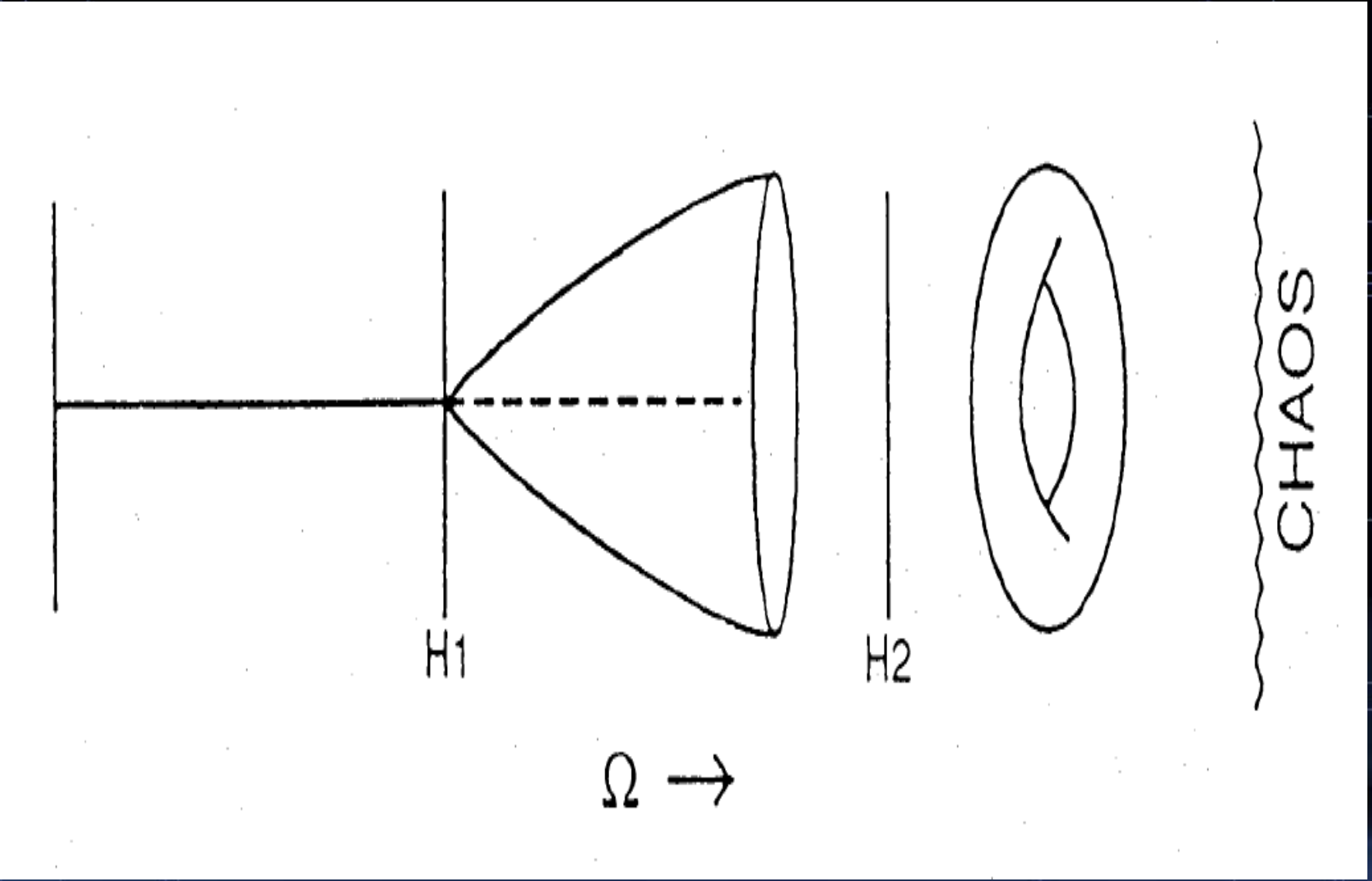
# Overview

- Background: What are stellar dynamos?
- Formulation of the model
  - Desirable Dynamics
  - Step by Step Formulation and Analysis
    - 1-D, 2-D
      - Formulation
      - Lorentz Force adds Hopf Bifurcations
      - Breaking degeneracy of 2$^{nd}$ Hopf
    - 3-D
      - Symmetry breaking
      - Chaos!
      - New breaking term for Reversibility
- Numerical Results
- Summary


Ohmic-Heating Magnet
Toroidal-Field Magnet
Poloidal-Field Magnet
Plasma
Magnetic Field Line

# Formulation of the Model



# Begin with 1D

- Simplify all hydrodynamical behavior of a star into a single variable, z
- We want to describe two steady convecting velocity fields, so we model z by:

$$\dot{z} = \mu - z^2$$

This gives rise to a saddle-node bifurcation, with fixed points:

$$P^+ : z = \sqrt{\mu} __ Stable$$

$$P^- : z = -\sqrt{\mu} _ Unstable$$

# Magnetic Field

- Toroidal field Bt = x
- Poloidal field Bp = y
- Set $q = x + iy = re^{i\Phi}$
- $r = (x^2 + y^2)^{0.5}$ Strength of magnetic field
- Now we have,

$$\dot{q} = (\lambda + i\omega)q + (a + ib)\lambda q - O(|q|^2 q)$$

# Magnetic Field cont.

- Using the definition of q, and reordering, we obtain the following system:

$$\dot{z} = \mu - z^2$$

$$\dot{x} = (\lambda + az)x - (\omega + bz)y$$

$$\dot{y} = (\lambda + az)y - (\omega + bz)x$$

And in cylindrical coordinates:

$$\dot{z} = \mu - z^2$$

$$\dot{r} = \lambda r + azr$$

$$\dot{\phi} = \omega - bz$$

# What do we have now?

$$\dot{z} = \mu - z^2$$
$$\dot{r} = \lambda r + azr$$
$$\dot{\phi} = \omega$$

$$J = \begin{bmatrix} -2z & 0 \\ ar & \lambda + az \end{bmatrix} \quad \begin{pmatrix} z \\ r \end{pmatrix}^*_{1/2} = \begin{pmatrix} \pm\sqrt{\mu} \\ 0 \end{pmatrix}$$

$$\lambda^e_{1/2} = \left\{ -2(\pm\sqrt{\mu}), \lambda + a(\pm\sqrt{\mu}) \right\}$$

$$T = \lambda + (a-2)(\pm\sqrt{\mu})$$

$$\Delta = -2\lambda(\pm\sqrt{\mu}) - 2a\mu$$

| | $\lambda < -a\sqrt{\mu}$ | $\lambda > -a\sqrt{\mu}$ |
|---|---|---|
| $P^+$ : | *Sink* | *Saddle* |
| | $\lambda < a\sqrt{\mu}$ | $\lambda > a\sqrt{\mu}$ |
| $P^-$ : | *Saddle* | *Source* |

$$T^2 - 4\Delta = \lambda^2 + (4 + 2\,\mathrm{a})\lambda\,(\pm\sqrt{\mu}) + (4\,\mathrm{a} + 4 + \mathrm{a}^2)\,\mu$$

# Lorentz Force

- Need to add back-reaction of magnetic field on the flow
- This force is proportional to B, so we add a term to z-dot (carefully):

$$\dot{z} = \mu - z^2 - (x^2 + y^2) \qquad \dot{z} = \mu - z^2 - r^2$$

$$\dot{x} = (\lambda + az)x - \omega y \qquad \dot{r} = \lambda r + azr$$

$$\dot{y} = (\lambda + az)y - \omega x \qquad \dot{\phi} = \omega$$

# What do we have now?

$$\boxed{\begin{aligned}\dot{z} &= \mu - z^2 - r^2 \\ \dot{r} &= \lambda r + azr\end{aligned}}$$

$$\begin{pmatrix} z \\ r \end{pmatrix}^*_{1/2} = \begin{pmatrix} \pm\sqrt{\mu} \\ 0 \end{pmatrix} \qquad J = \begin{bmatrix} -2z & -2r \\ ar & \lambda + az \end{bmatrix}$$

$$\begin{pmatrix} z \\ r \end{pmatrix}^*_{3} = \begin{pmatrix} -\dfrac{\lambda}{a} \\ \sqrt{\mu + \dfrac{\lambda^2}{a^2}} \end{pmatrix}$$

$$J(z,r)^*_3 = \begin{bmatrix} \dfrac{2\lambda}{a} & -2\sqrt{\mu + \dfrac{\lambda^2}{a^2}} \\ a\sqrt{\mu + \dfrac{\lambda^2}{a^2}} & 0 \end{bmatrix}$$

$$|\lambda| > a^2\sqrt{\mu} \rightarrow \Delta < 0 \rightarrow Saddle$$

$$\boxed{\begin{array}{ll} \multicolumn{2}{l}{|\lambda| < a^2\sqrt{\mu} \rightarrow \Delta > 0} \\ \hline \lambda > 0 & Unstable \\ \lambda = 0 & Center \\ \lambda < 0 & Stable \end{array}}$$

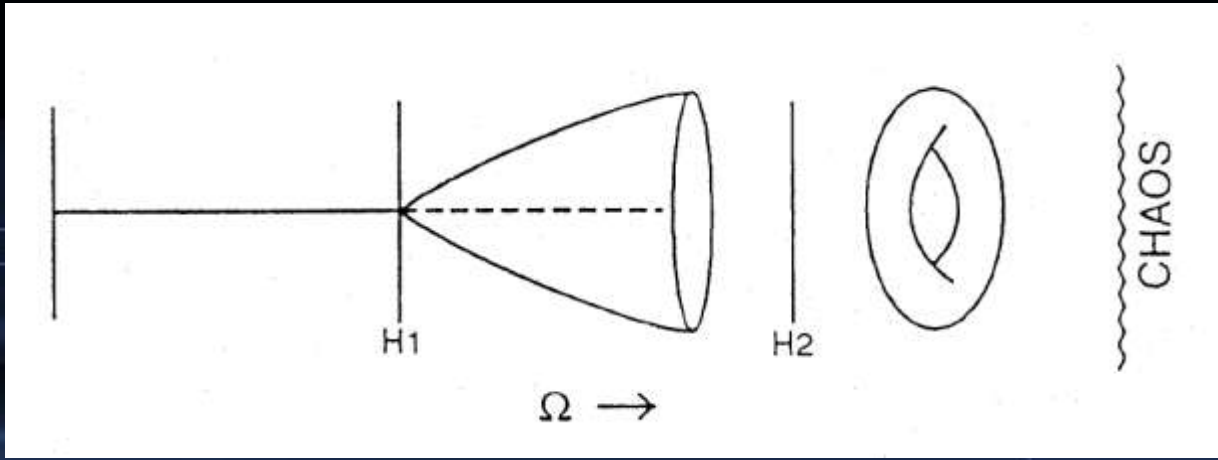
$$T = \frac{2\lambda}{a} > 0 \qquad \Delta = 2a(\mu - \frac{\lambda^2}{a^2})$$

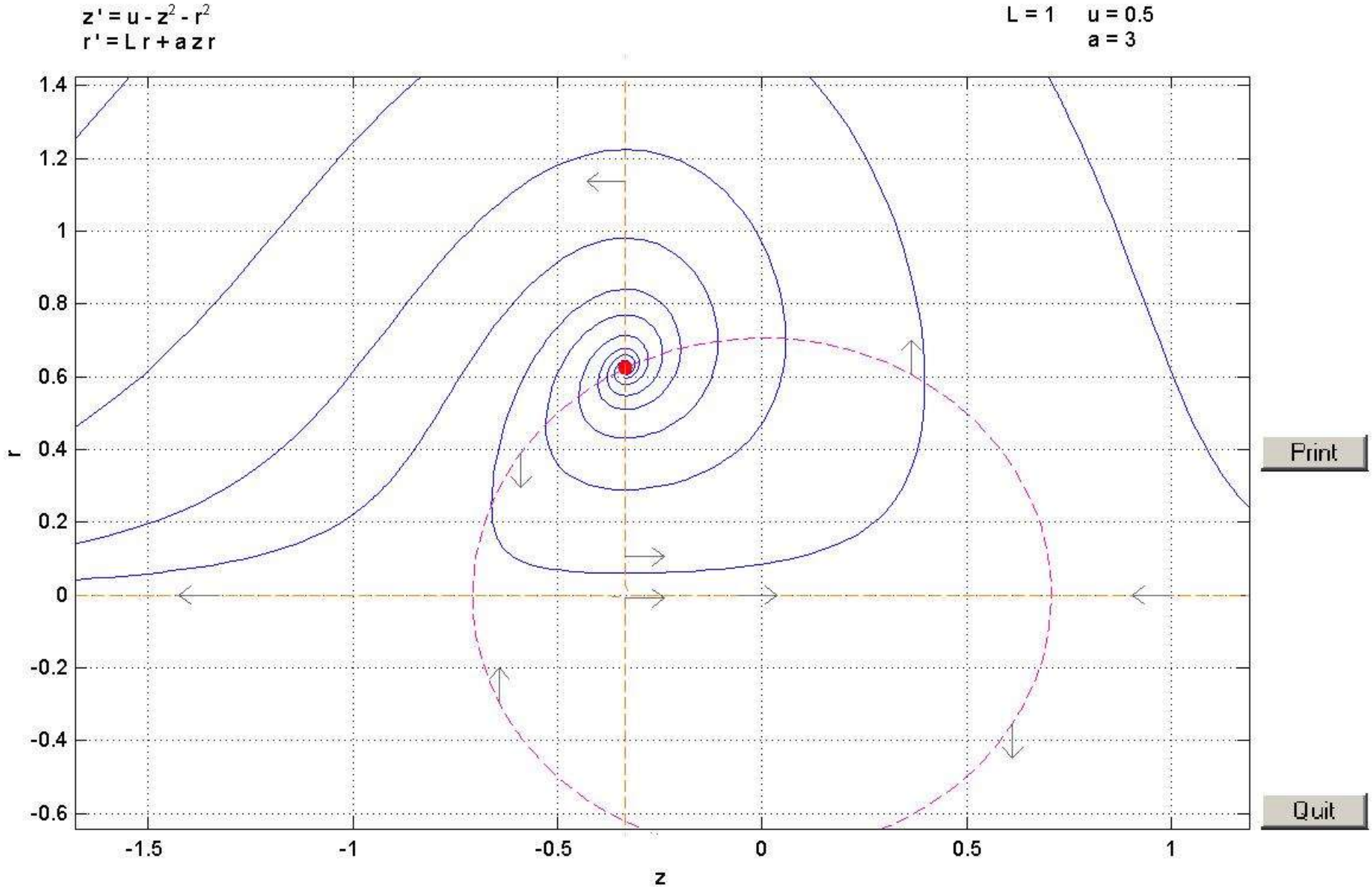
$$T^2 - 4\Delta = \frac{4\lambda^2}{a^2} - 8\mathrm{a}(\mu - \frac{\lambda^2}{a^2})$$

$$\boxed{-\sqrt{\frac{2a^3\mu}{1+2a}} < \lambda < \sqrt{\frac{2a^3\mu}{1+2a}} \quad Spirals}$$

# Quick Reality Check

- We are analyzing r vs. z
- Fixed point in r ($r \neq 0$) means? Periodic orbit in x and y!
- Periodic orbit in r means?
- Toroidal orbit in x and y (and z)!






The backward orbit from (-0.95, 0.26) --> a possible eq. pt. near (-0.33, 0.65).
Ready.
Pick a center and drag the mouse, or just click on a center.
Computing the field elements.
Ready.

# Breaking Degeneracy

- We want a torris that will break into chaos, so first we need a viable torris that is maintained in parameter space!
- To do this, a cubic term is added to z-dot, breaking the symmetry that caused the degeneracy.
- Now our system: (c<0)

$$\dot{z} = \mu - z^2 - r(x^2 + y^2) + cz^3$$

$$\dot{x} = (\lambda + az)x - \omega y$$

$$\dot{y} = (\omega\lambda + az)y - \omega x$$

# *What do we have now?(1)*

- New fixed point, and total remap of three old ones.
- No degeneracy
- Heteroclinic Connection
- Stable, unique toroidal orbits, shown as limit cycles in r-z plane

# What do we have now? (2)

$$\boxed{\begin{aligned}\dot{z} &= \mu - z^2 - r^2 + cz^3 \\ \dot{r} &= \lambda r + azr\end{aligned}}$$

$$\begin{pmatrix} z \\ r \end{pmatrix}_4^* = \begin{pmatrix} -\dfrac{\lambda}{a} \\ \sqrt{\mu - \dfrac{\lambda^2}{a^2} - c\dfrac{\lambda^3}{a^3}} \end{pmatrix} \qquad J = \begin{bmatrix} -2z + 3cz^2 & -2r \\ ar & \lambda + az \end{bmatrix}$$

$$J(z,r)_4 = \begin{bmatrix} 2\dfrac{\lambda}{a} + 3c\dfrac{\lambda^2}{a^2} & -2\sqrt{\mu - \dfrac{\lambda^2}{a^2} - c\dfrac{\lambda^3}{a^3}} \\ a\sqrt{\mu - \dfrac{\lambda^2}{a^2} - c\dfrac{\lambda^3}{a^3}} & 0 \end{bmatrix} \qquad \begin{pmatrix} z \\ r \end{pmatrix}_{1,2,3}^* = \begin{pmatrix} z_{1,2,3} : \{cz^3 + z^2 - \mu = 0\} \\ 0 \end{pmatrix}$$

$$T = 2\frac{\lambda}{a} + 3c\frac{\lambda^2}{a^2} \rightarrow$$

$$\Delta = \frac{2}{a^2}(a^3\mu - a\lambda^2 - c\lambda^3)$$

$$\lambda = 0 \rightarrow \qquad \Delta = 2a\mu \qquad \mu > 0 \rightarrow \qquad Hopf$$

$$\lambda = -\frac{2a}{3c} \rightarrow \quad \Delta = 2a\mu - \frac{8}{27}\frac{a}{c^2} \quad \mu > \frac{4}{27c^2} \rightarrow \quad Hopf$$

$$T^2 - 4\Delta = 9c^2\frac{\lambda^4}{a^4} + \left(\frac{8}{a^2c} + \frac{12}{a^3c}\right)\lambda^3 + \left(\frac{4}{a^2} + \frac{8}{a}\right)\lambda^2 - 8a\mu$$

Z1 =
[1/9*(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3)+100/9/(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3)+10/9]

Z2 =

[ -1/18*(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3)-50/9/(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3)+10/9+1/2*i*3^(1/2)*(1/9*(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3)-100/9/(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3))]

Z3 =

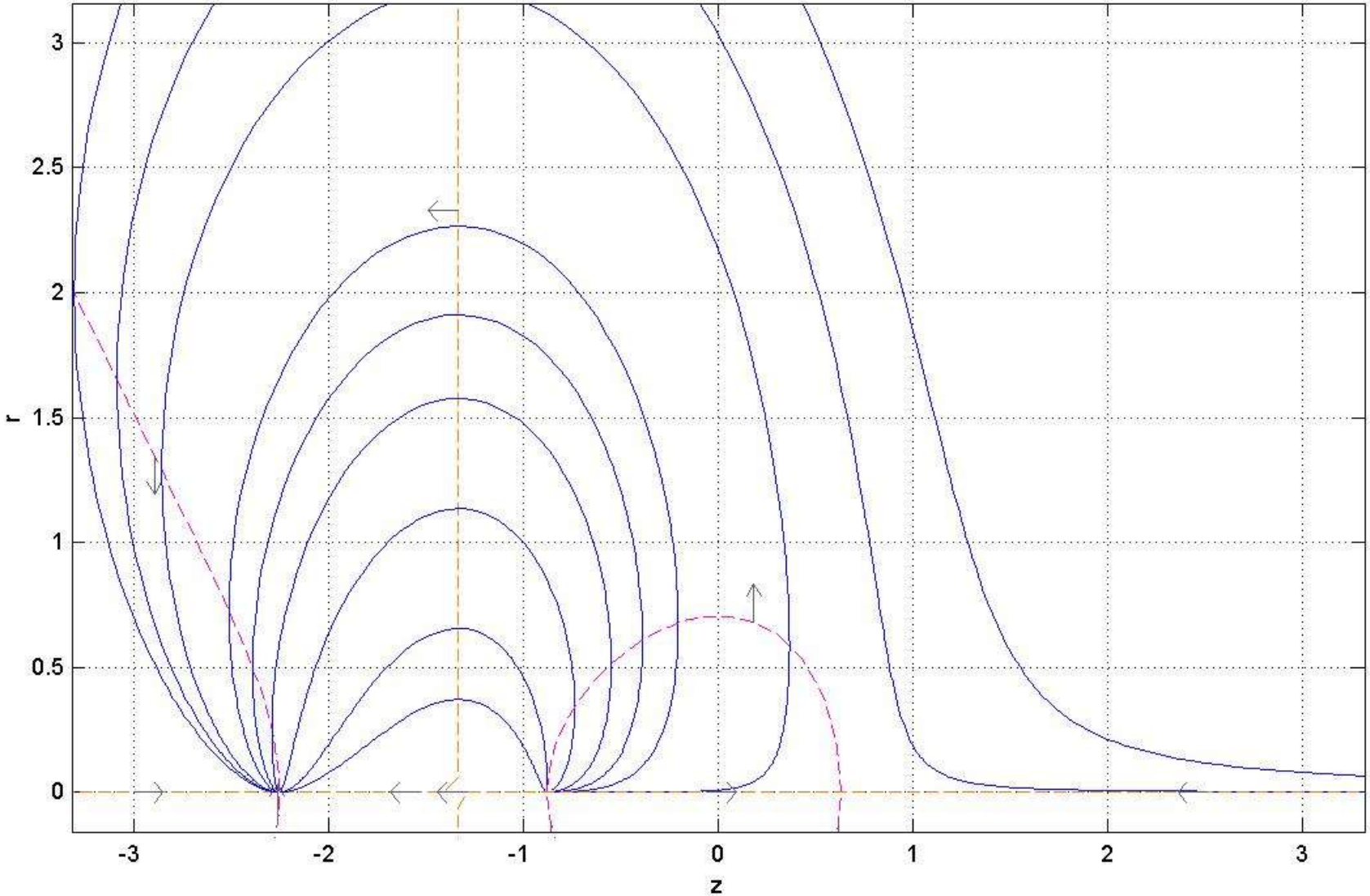
[ -1/18*(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3)-50/9/(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3)+10/9-1/2*i*3^(1/2)*(1/9*(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3)-100/9/(1000-1215*u+45*(-1200*u+729*u^2)^(1/2))^(1/3))]

Lambda1 =
[1/2/a^2*(2*lam*a+3*c*lam^2+(4*lam^2*a^2+12*lam^3*a*c+9*c^2*lam^4+8*a^3*lam^2+8*a^2*c*lam^3-8*a^5*u)^(1/2))]

Lambda2 =
[ 1/2*(2*lam*a+3*c*lam^2-(4*lam^2*a^2+12*lam^3*a*c+9*c^2*lam^4+8*a^3*lam^2+8*a^2*c*lam^3-8*a^5*u)^(1/2))/a^2]

$z' = u - z^2 - r^2 + c z^3$
$r' = L r + a z r$

$L = 4$ $u = 0.5$
$a = 3$ $c = -0.4$




Print

Quit

The backward orbit from (1.3, 1.1) left the computation window.
Ready.
The forward orbit from (0.82, 0.8) --> a possible eq. pt. near (-2.3, 0.0011).

Wi... Mi... <St... c... Address http://www.hotmail.com/ Go EN

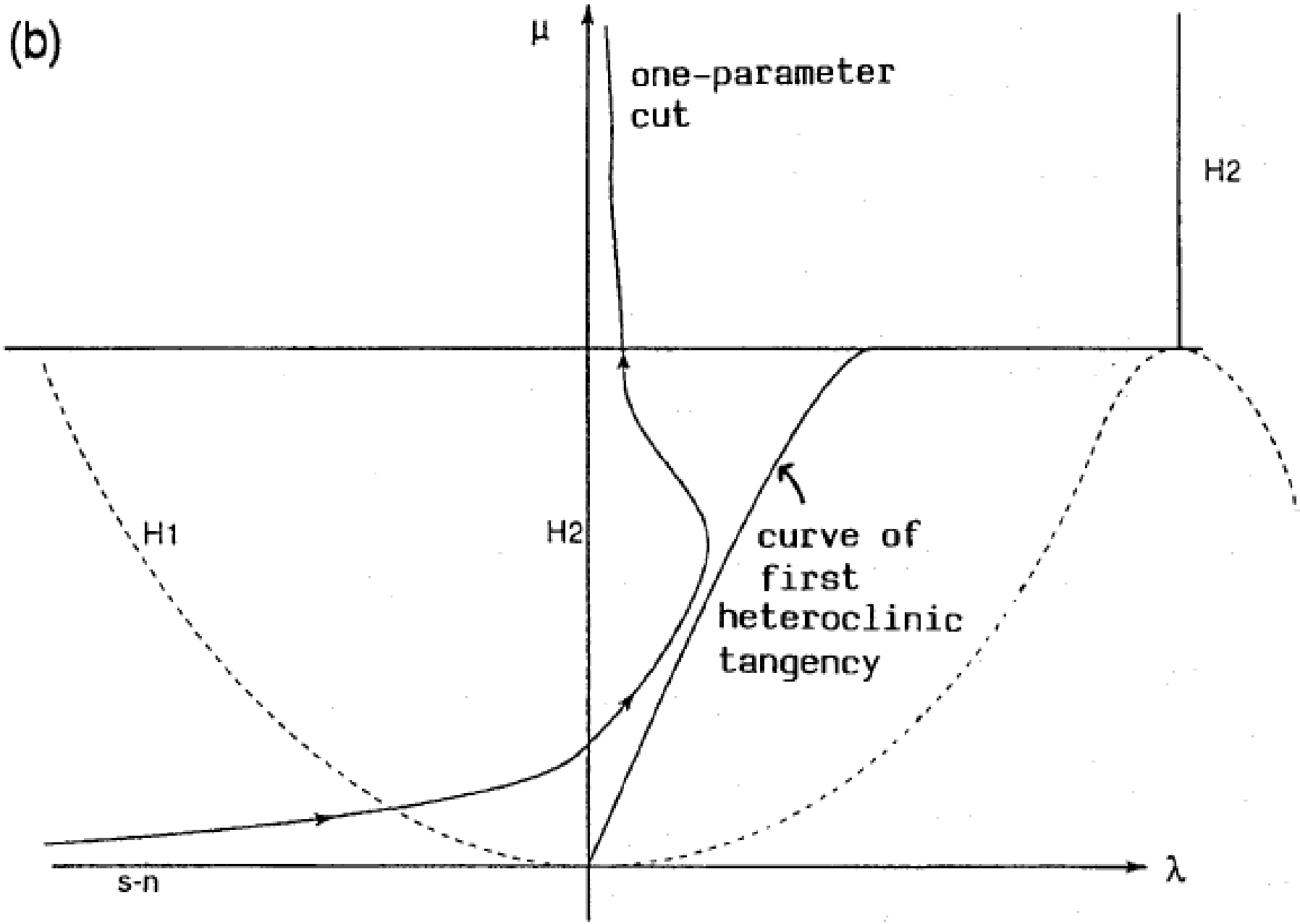
# What about Chaos?

- Since system is essentially 2-D, no chaos possible.
- To break axisymmetric property of system, we add cubic term to toroidal field (x).
- Finally(?), our system:

$$\dot{z} = \mu z - z^2 - r(x^2 + y^2) + cz^3$$

$$\dot{x} = (\lambda + az)x + r\omega y - d(x^2 + y^2)$$

$$\dot{y} = (\lambda - az)y - r\omega x$$


(b)
μ
one-parameter
cut
H2
H1
H2
curve of
first
heteroclinic
tangency
s-n
λ

# As if this Wasn't Enough!

- After 10 years, an improvement has been made on the model.
- We last left our model after adding a symmetry-breaking cubic term to x
- This unfortunately breaks the y->-y, x->-x reversibility of the system
- Another possibility for achieving the same result, without losing reversibility is:

$$\dot{z} = \mu - z^2 - (x^2 + y^2) + cz^3$$

$$\dot{x} = (\lambda + az)x - \omega y + d(3x^2 - y^2)$$

$$\dot{y} = (\lambda + az)y - \omega x + d(3x^2 y - y^3)$$

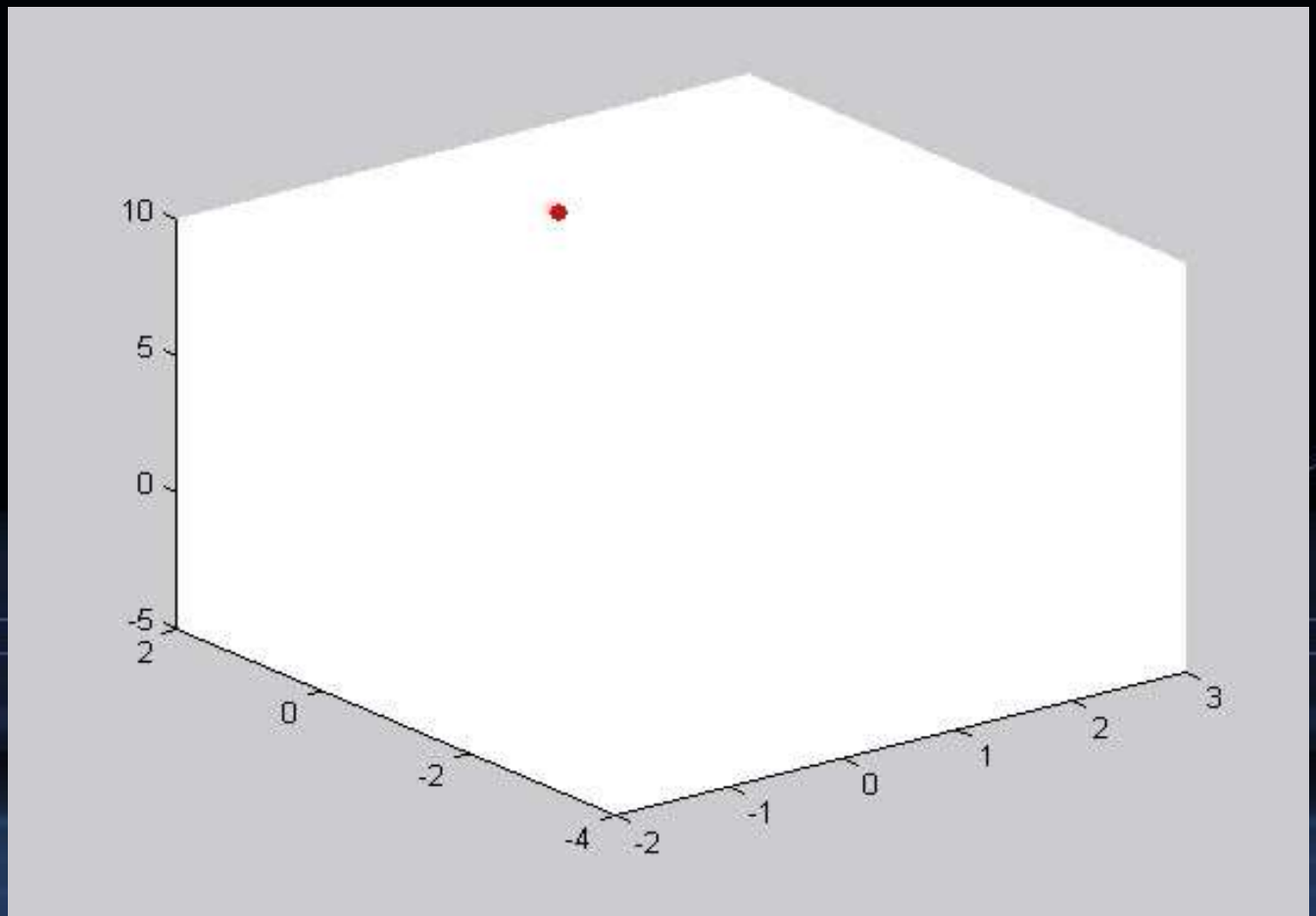
$$\dot{z} = \mu - z^2 - r^2 + cz^3$$

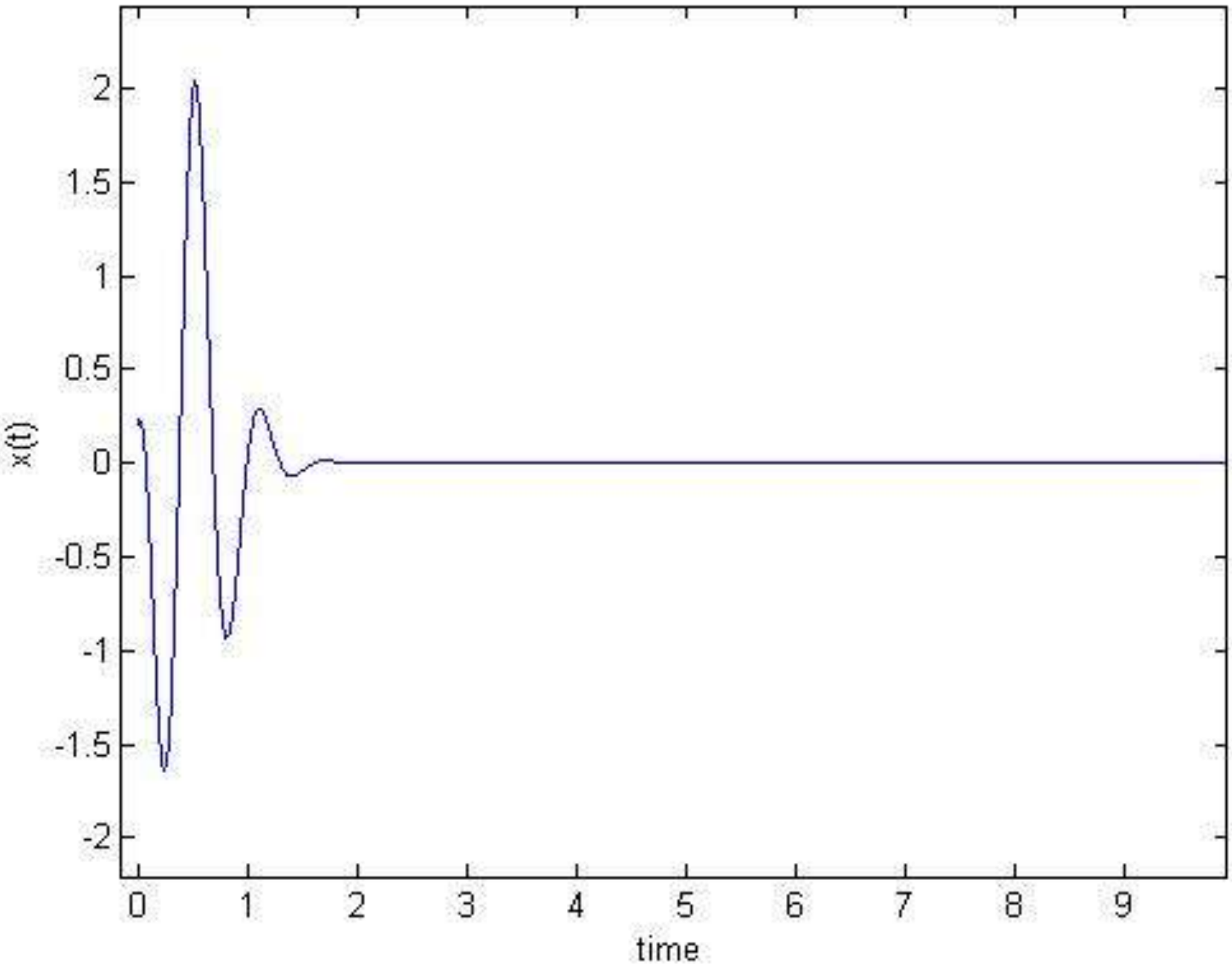
$$\dot{r} = \lambda r + azr + dr^3 \cos 2\phi$$

$$\dot{\phi} = \omega + drz^2 \sin 2\phi$$

# Numerical Results

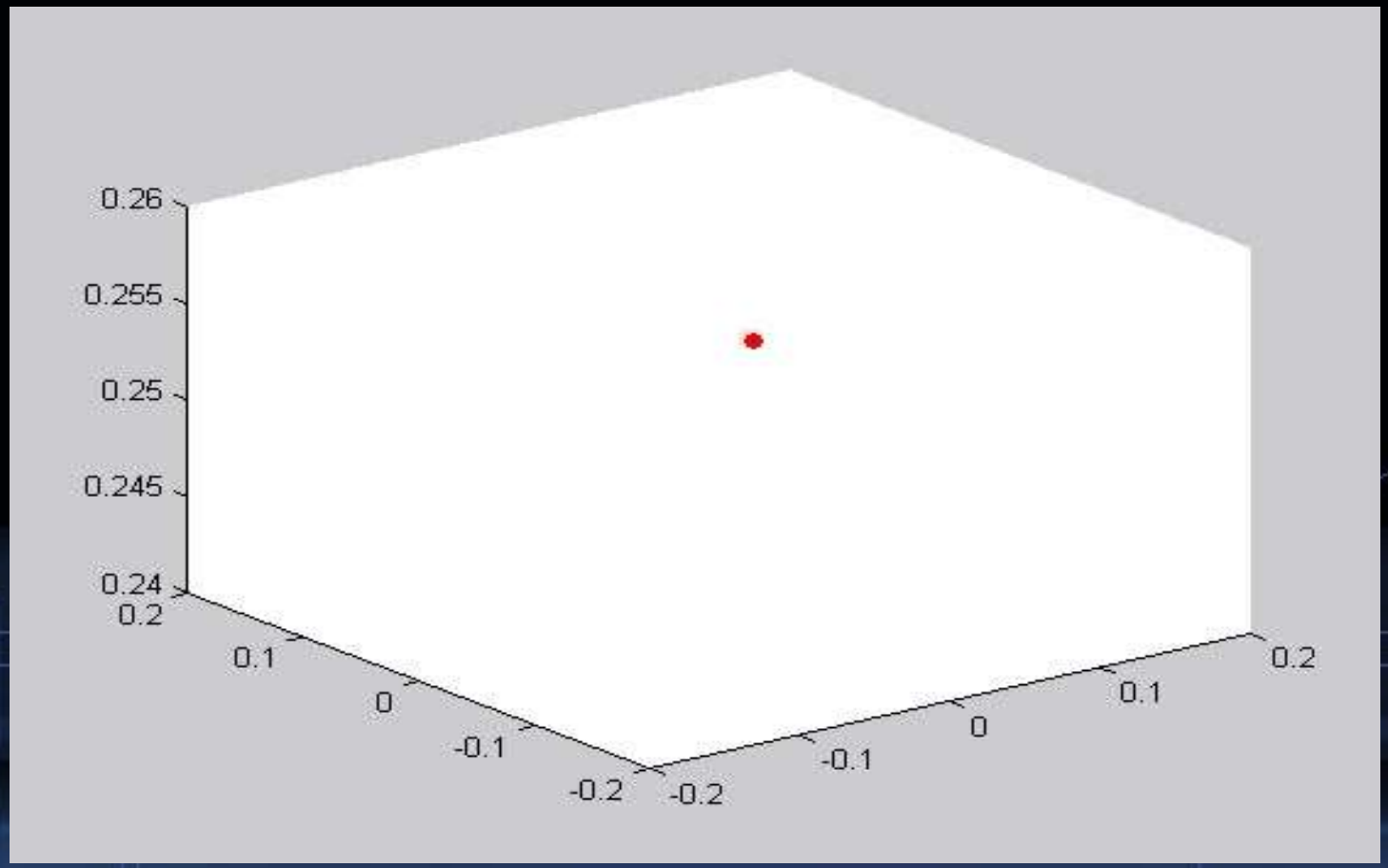
- 1 $\Omega << 1$ - All trajectories collapsing to the fixed point P+.

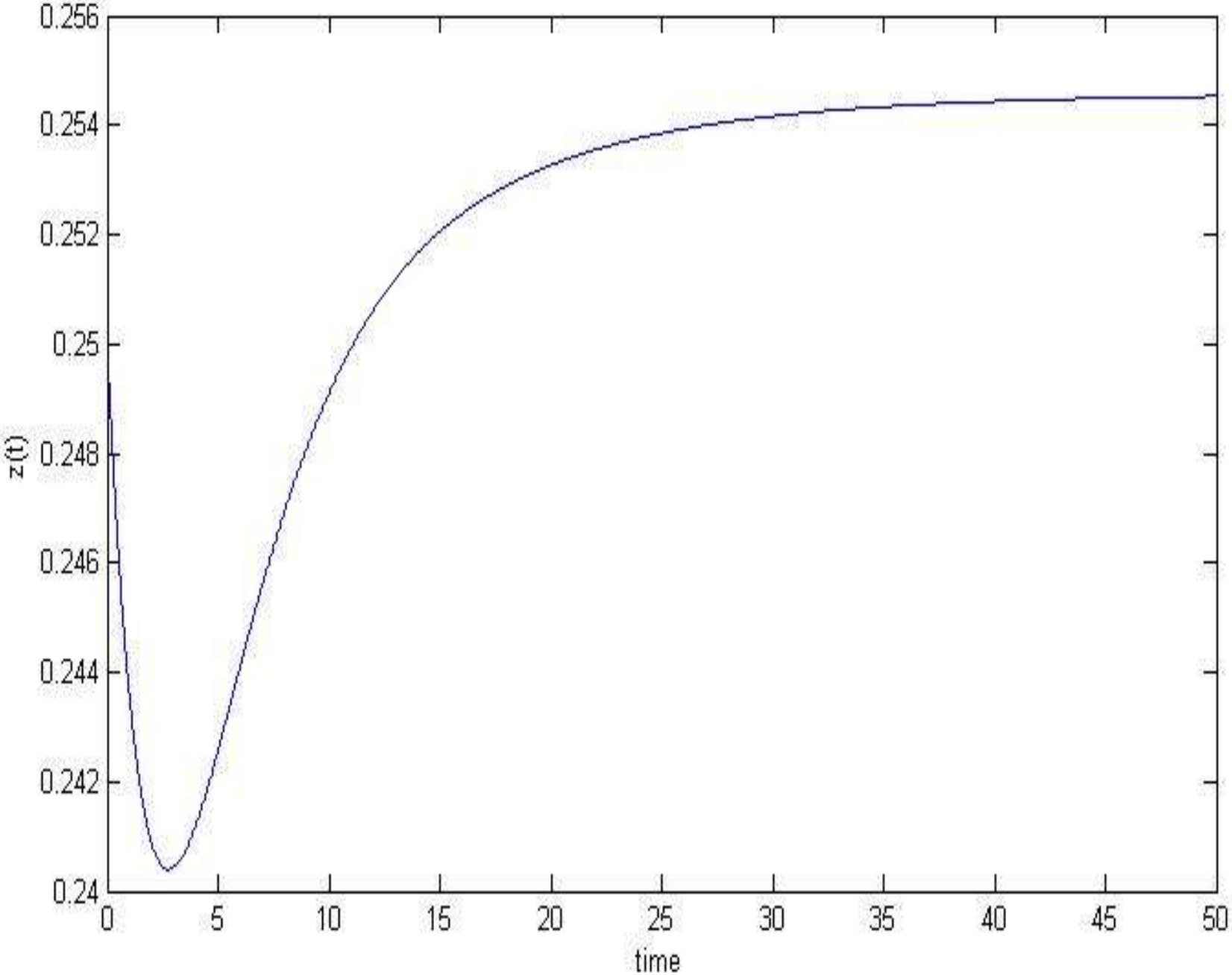



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
x(t)
0
1
2
3
4
5
6
7
8
9
time

# Numerical Results

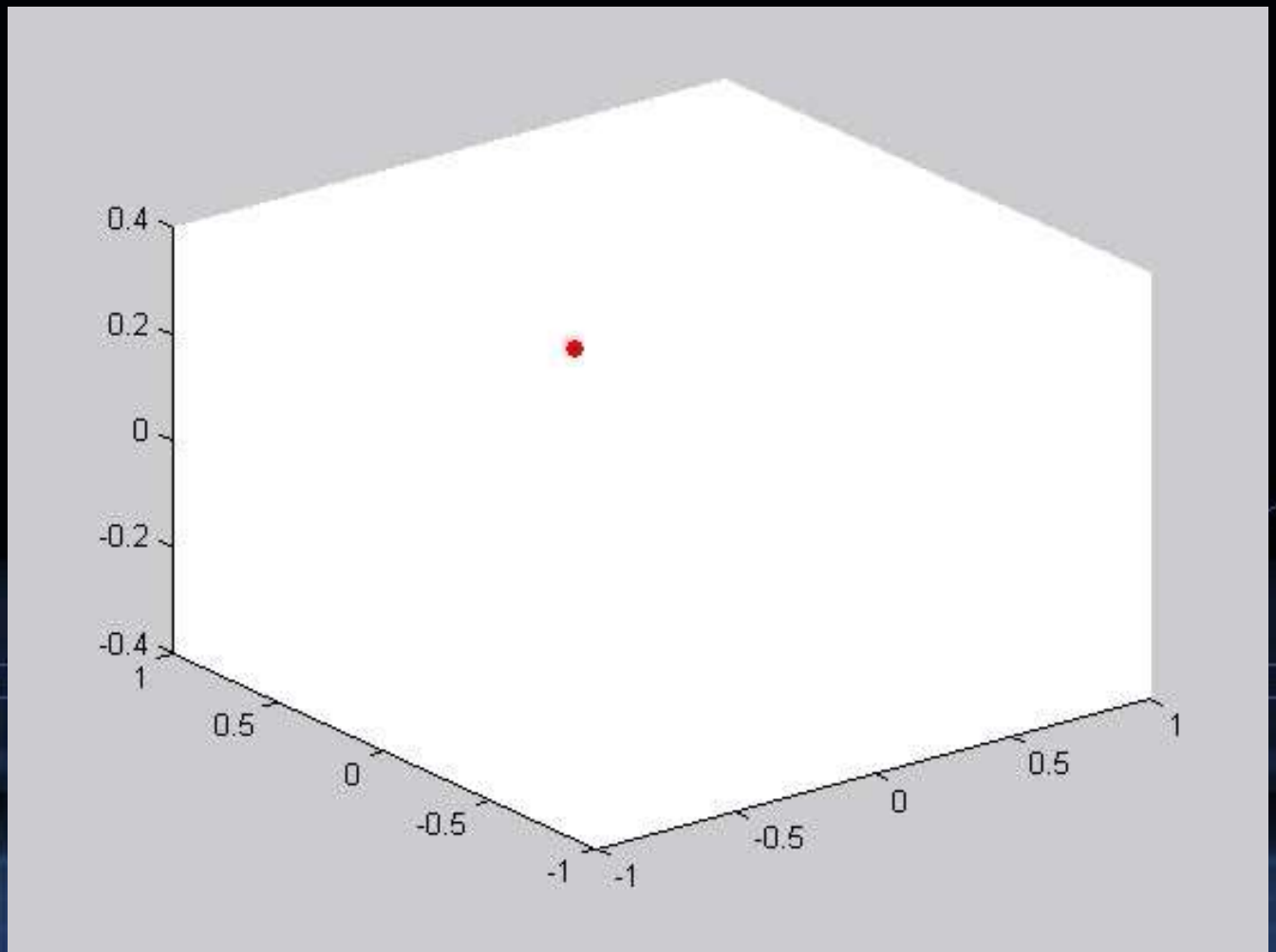
- 2 $\Omega = 6.38 \times 10^{-3}$ - First Hopf bifurcation.




0.256
0.254
0.252
0.25
0.248
0.246
0.244
0.242
0.24
z(t)
0
5
10
15
20
25
30
35
40
45
50
time

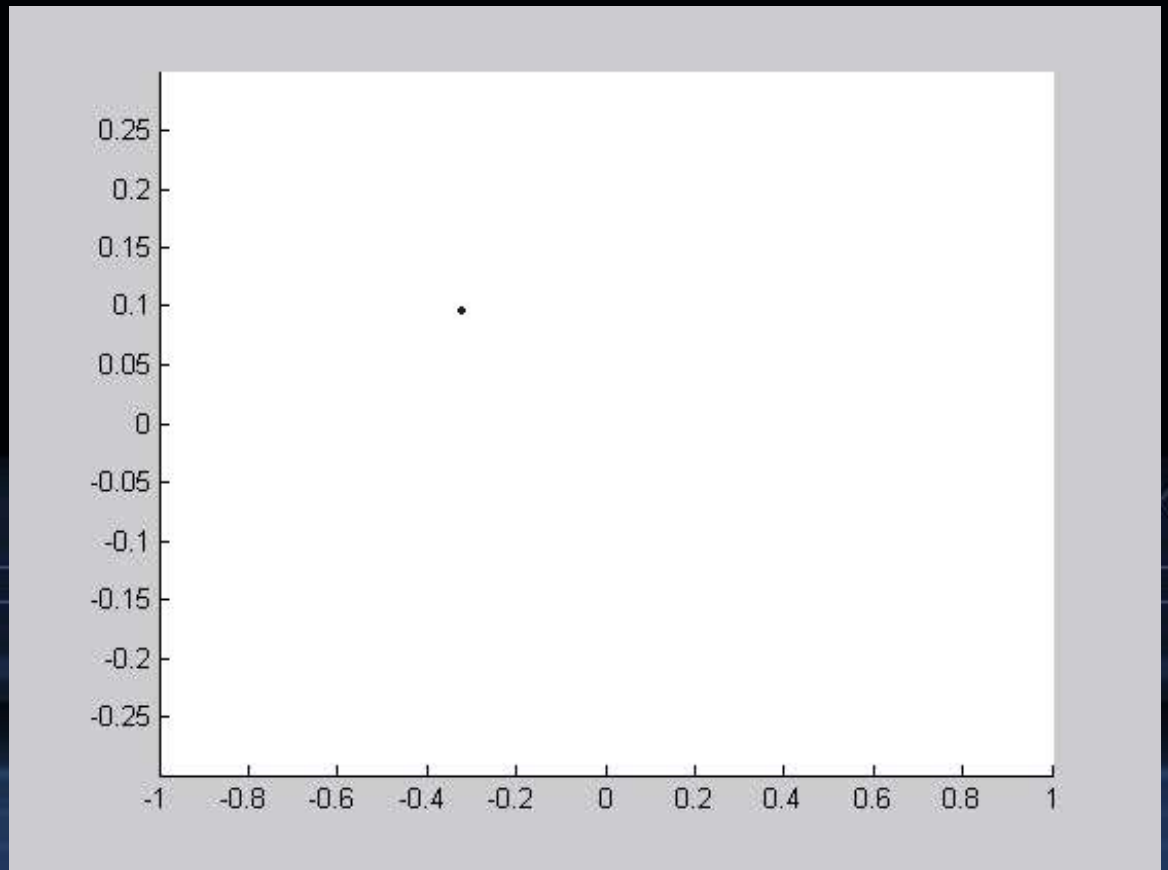
# Numerical Results

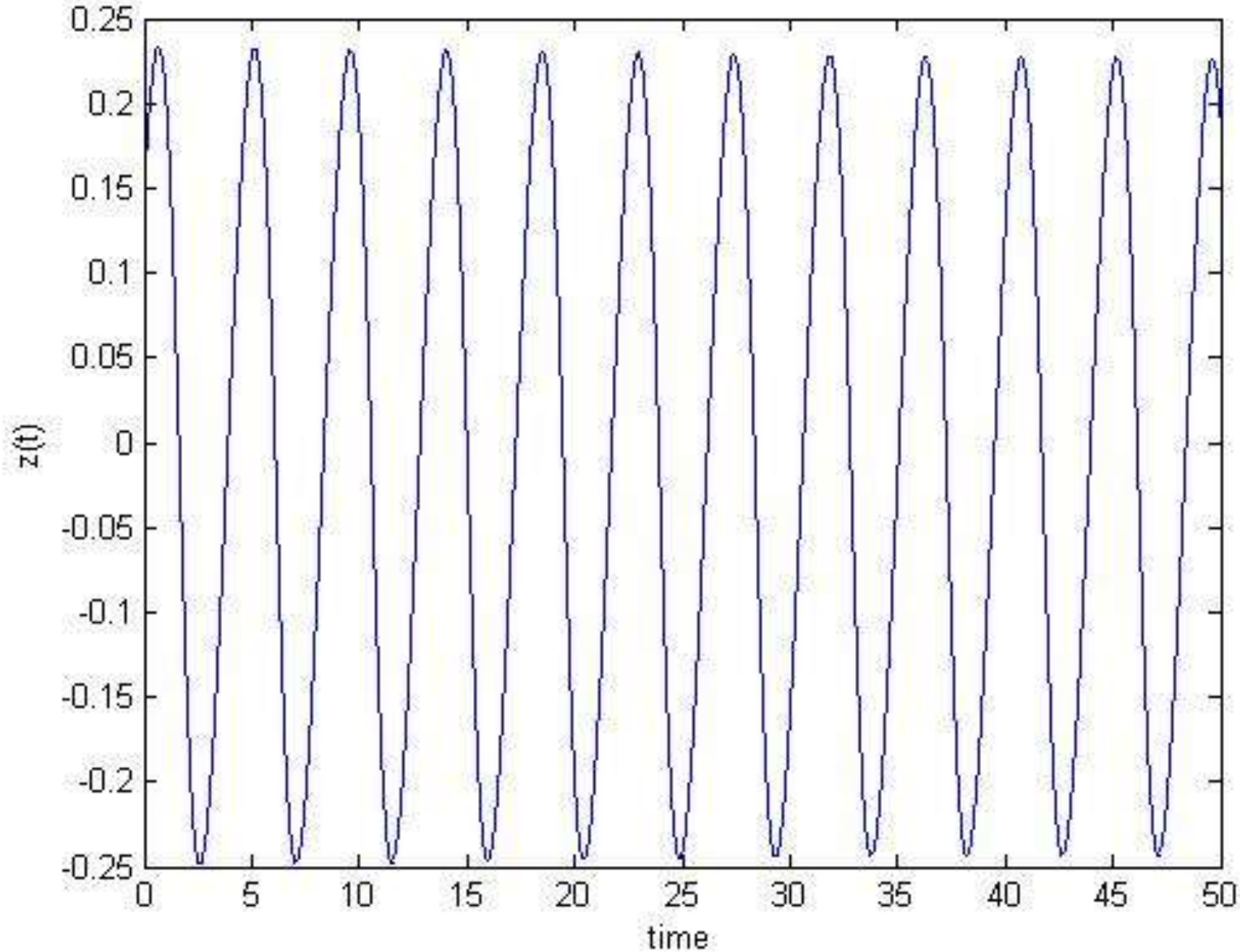
- 3 $\Omega = 6.38 \times 10^{-3}$ - Second Hopf bifurcation.



# Numerical Results

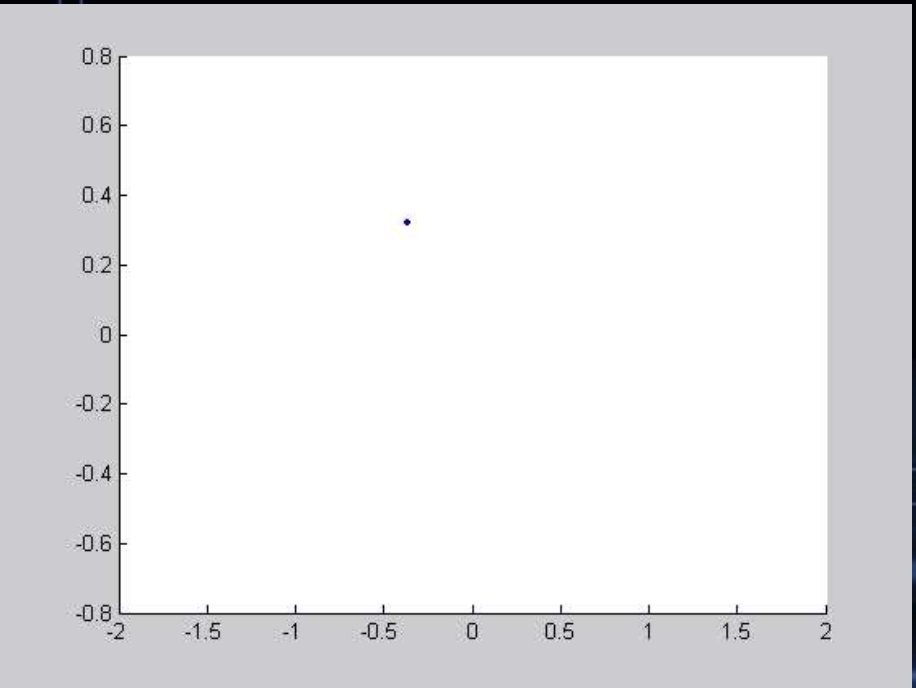
- 3 $\Omega = 6.38 \times 10^{-3}$ - Second Hopf bifurcation.
- Poincaré Plane

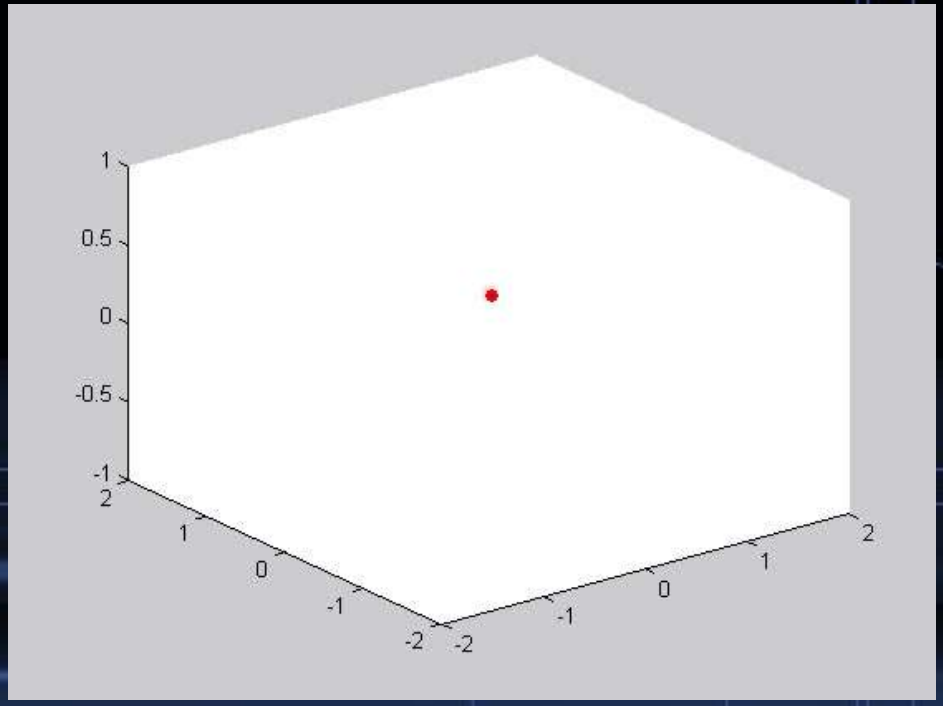


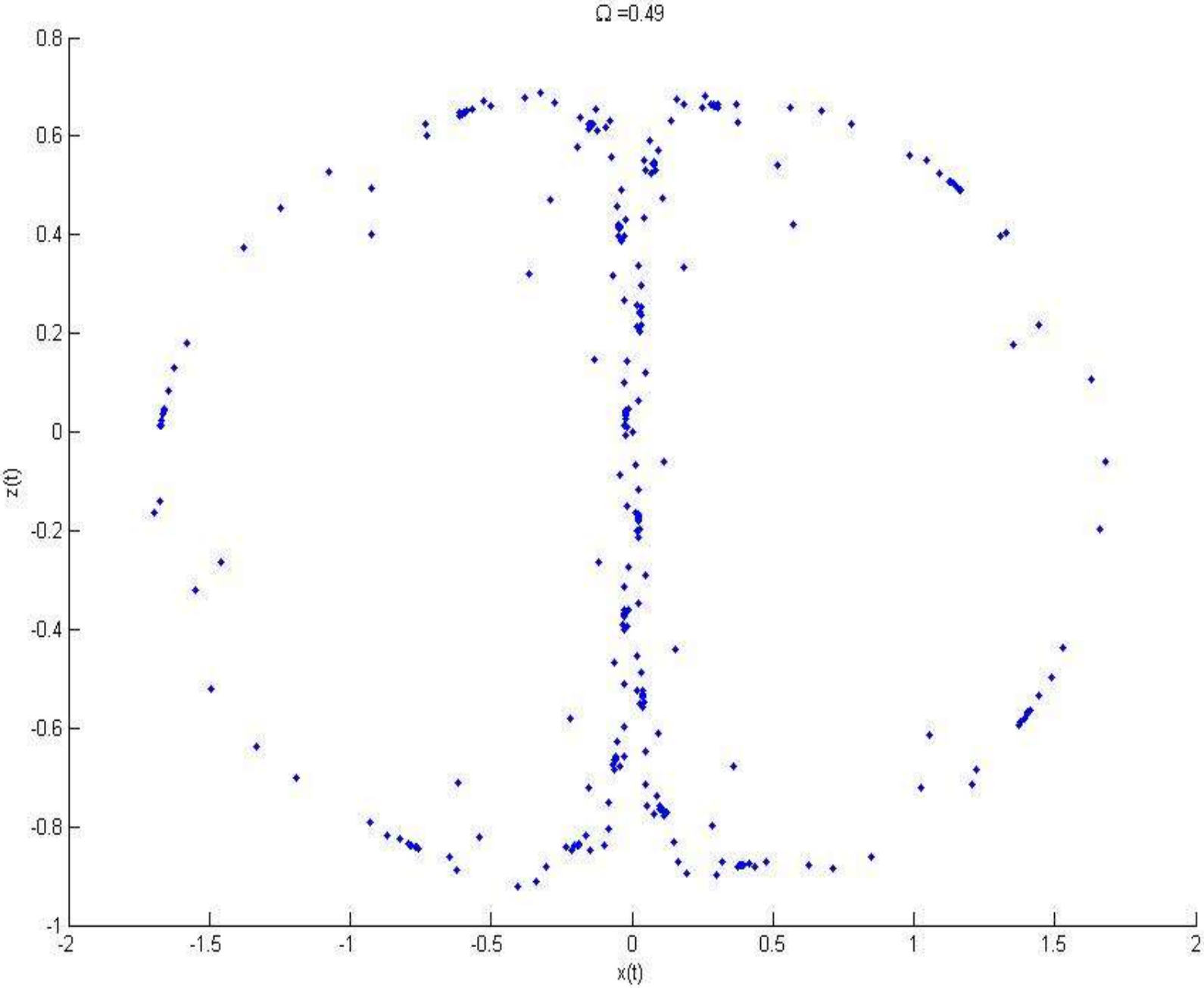

0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
z(t)
0
5
10
15
20
25
30
35
40
45
50
time

# Numerical Results

- 4 $\Omega = 0.49$ - Torus folding onto a chaotic attractor.

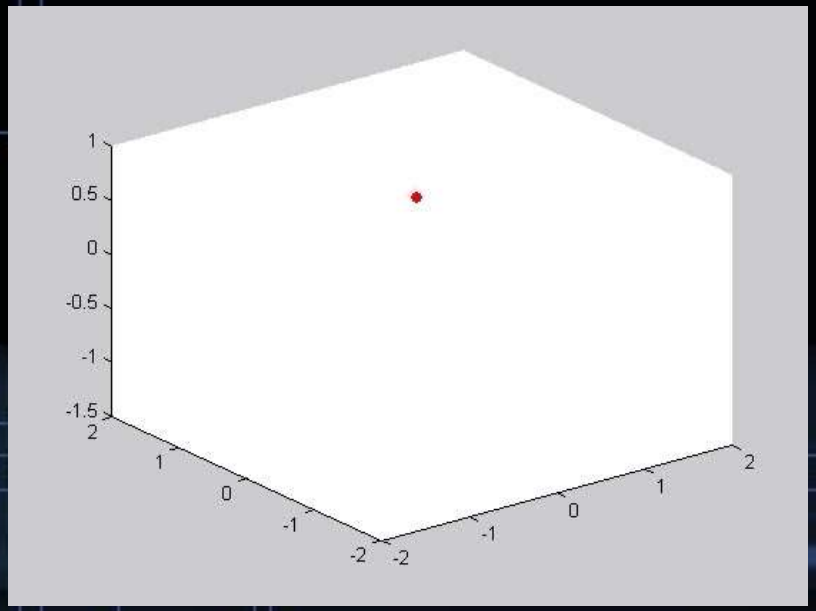


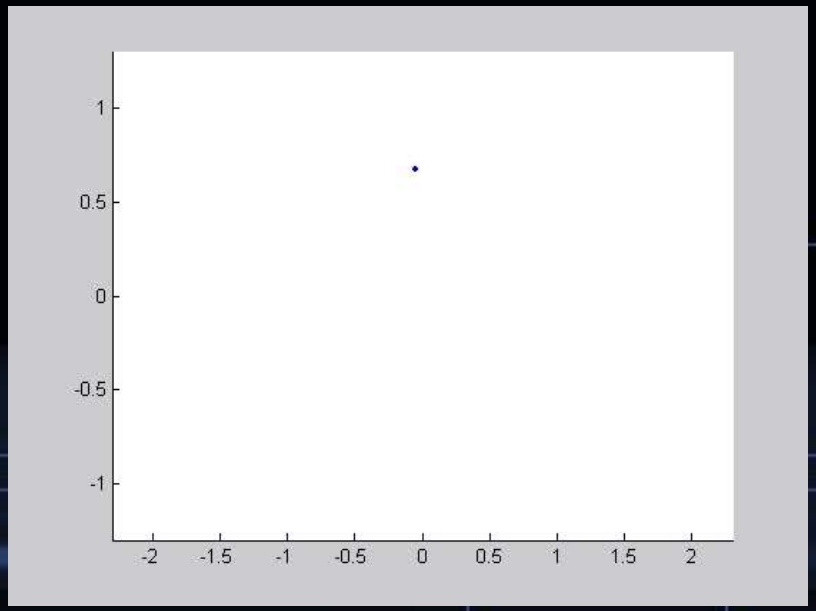



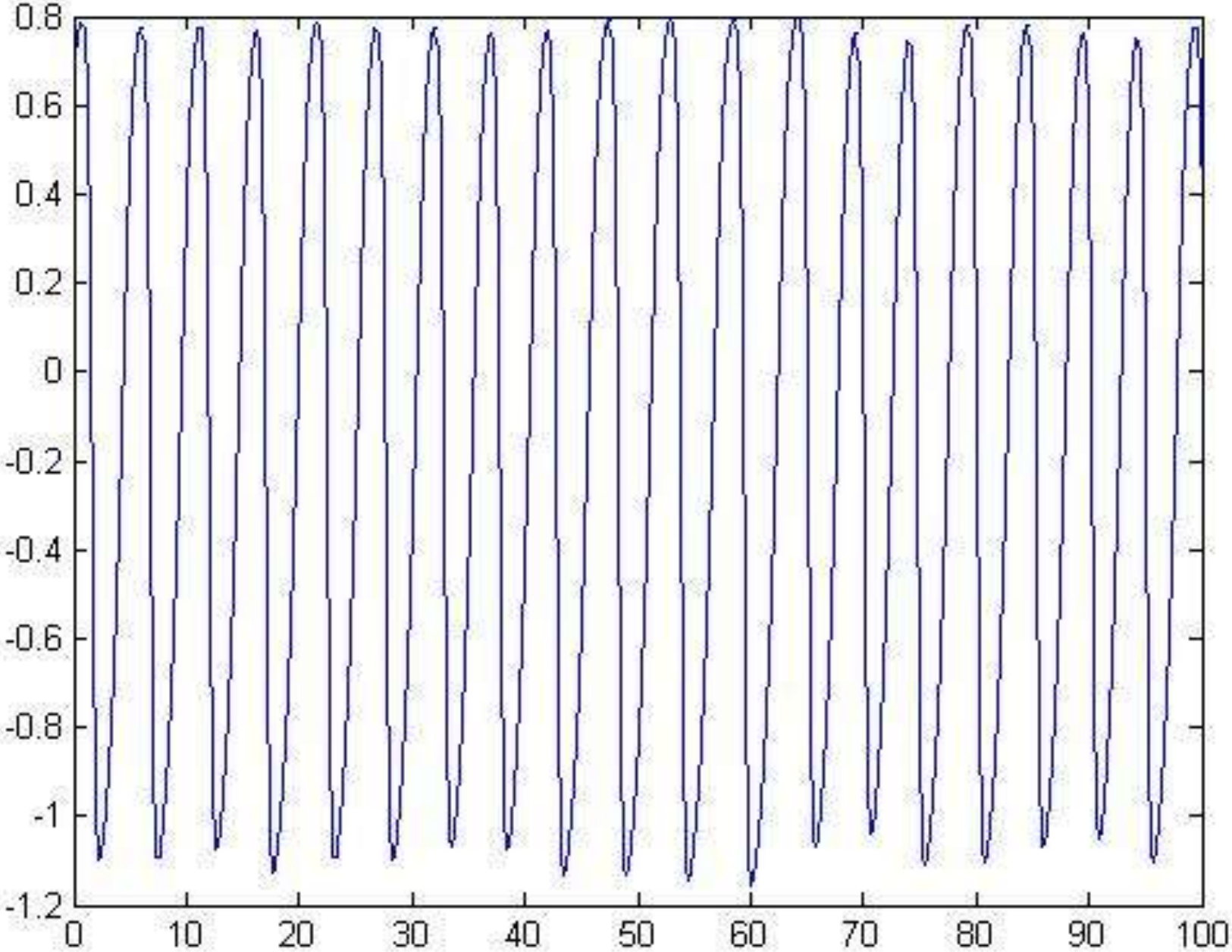

Ω =0.49
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
z(t)
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x(t)

# Numerical Results

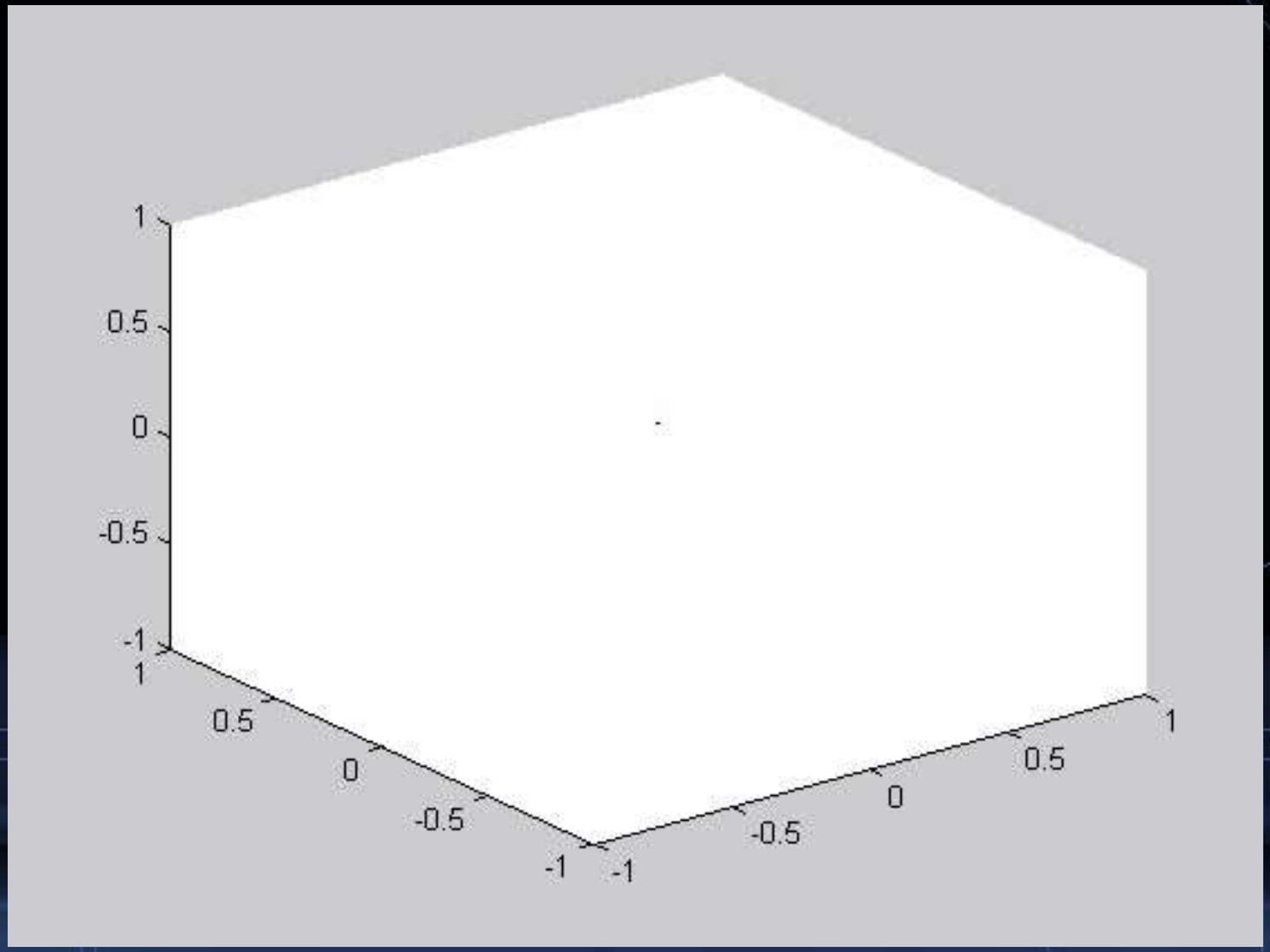
- 4 $\Omega = 0.85$ - Torus folding onto a chaotic attractor.




0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
0
10
20
30
40
50
60
70
80
90
100

# Bifucartions numerically



# Summary

- We re-derived the model found in the paper, and in addition we did a detailed analysis of the bifurcations occurring in the system.
- We could see that the model is fairly successful reproducing the different qualitative regimes of magnetic activity in the star.
- Even being an artificial model, it might be very helpful to understand the processes occurring in such complex system.
- Furthermore is a very rich  non-linear model in which a great number of features.