San Diego State University

M 542 Intro. to Numerical Solutions to D.E

Final Project

Solution to a Hyperbolic problem using Mimetic Methods

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Outline

- Preliminaries
- Discrete Operators: D &G.
- 1-D problem. Results.
- 2-D Hyperbolic problem. Results.
- Conclusions.
- References.

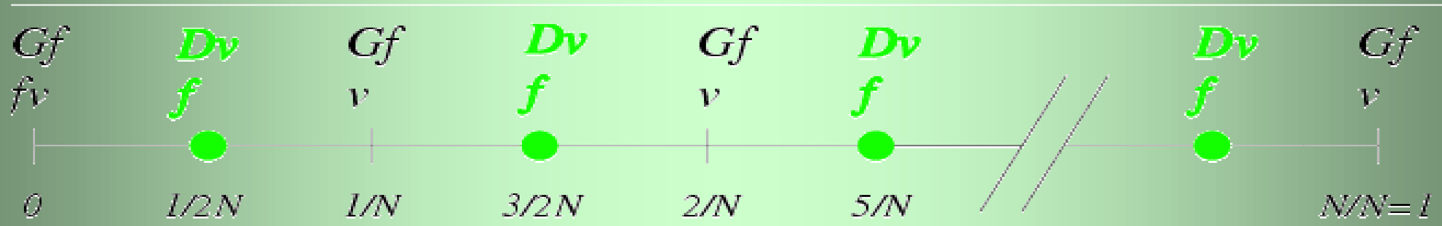
Preliminaries: Mimetic Methods.

Mimetic methods are a set of discrete operators DIV, GRAD, CURL that approximate continuum differential operators ***div, grad, curl***, and preserve fundamental properties established in vector calculus.

Construction (Castillo and Grone, 2003):

The central problem is to find DIV, and GRAD to satisfy a discrete analogue of the divergence theorem:

$$\int_{\Omega} div(v)\, f\, dV + \int_{\Omega} v . grad(f)\, dV = \int_{\partial\Omega} f\, v . \hat{n}\, dS$$



A collection of values of ***v*** and *f* are defined in a staggered grid and collected in vectors **v** and **f**. Operators DIV, and GRAD become matrices D, and G that satisfy

$$< \hat{D}\mathbf{v},\mathbf{f} > + < \mathbf{v}, G\,\mathbf{f} > = < B\,\mathbf{v}, \mathbf{f} >$$

Second Order Discrete Operators

The simplest discrete divergence is given by

$$(Dv)_{i+\frac{1}{2}} = \frac{v_{i+1} - v_i}{h} \qquad 0 \le i \le N\text{-}1$$

$$\boldsymbol{hD} = \begin{bmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix} \in R^{N\times(N+1)}$$

- The discrete gradient is defined $0 \le i \le N-1$

$$(Gf)_i = \frac{f_{i+1/2} - f_{i-1/2}}{h},$$

- $hG = \begin{bmatrix} \frac{-8}{3} & 3 & \frac{-1}{3} & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & 0 & -1 & 1 & 0 \\ 0 & \cdots & 0 & \frac{1}{3} & -3 & \frac{8}{3} \end{bmatrix} \in R^{(N+1)\times(N+2)}$

The discrete Laplacian is $L = \hat{D}G$

$$where \hat{D} = \begin{bmatrix} 0 \\ D \\ 0 \end{bmatrix} \in R^{(N+2)\times(N+2)}$$

Castillo and Grone (2003) introduced matrix *B* to write the discrete identity using standard inner products,

$$< D\,\mathbf{v},\mathbf{f} > + <\mathbf{v},\, G\,\mathbf{f} > = < B\,\mathbf{v},\, f >$$

$$B = \begin{bmatrix} -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

In addition, they proposed matrix *B*' by using weighted inner products,

$$\left\langle \hat{D}v,\, f_{cb} \right\rangle_Q + \left\langle G^T v,\, f_{cb} \right\rangle_P = \left\langle B'v,\, f_{cb} \right\rangle$$

where *Q* $\in R^{(N+2)\times(N+2)}$ and P $\in R^{(N+1)\times(N+1)}$ are positive definite matrices consistently built. In this case,

$$B' = D + (PG)^T$$

Model Problem

- General elliptic partial differential equation:

$$-div\left(K\, grad\, u\right) = F(x) \ , \quad x \in \Omega$$

K is a tensor function, *F(x)* is a source term

- Robin boundary conditions:

$$\beta\left(\hat{n}, K\, grad\, u\right) + \alpha\, u = \gamma \ , \quad x \in \partial\Omega$$

α , β, γ are function given on $\partial\Omega$

1D cases of study

Case 1:

Let, $F(x) = \dfrac{-\lambda^2 e^{\lambda x}}{e^{\lambda} - 1}$

- The elliptic partial differential equation is:

$$-\nabla^2 u(x) = F(x) \quad \text{on } [0, 1]$$

- Robin boundary conditions:

$\alpha f(0) - \beta f'(0) = -1$

$\alpha f(1) + \beta f'(1) = 0$

$$\alpha = -e^{\lambda}$$

$$\beta = \frac{(e^{\lambda} - 1)}{\lambda}$$

The exact solution is:

$$u(x) = \frac{e^{\lambda x} - 1}{e^{\lambda} - 1}$$

- The discretization process leads to any of these two systems of linear equations :

$$(\alpha A + \beta BG + L)\tilde{u} = f, \quad (1)$$

$$A = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \in R^{(N+2)\times(N+2)}$$

$$(\alpha A + \beta B' G + L)\tilde{u} = f, \quad (2)$$

for $f = (-1, F(x_{\frac{1}{2}}), F(x_{\frac{3}{2}}), \ldots\ldots\ldots, F(x_{n-\frac{1}{2}}), 0)^T$

Numerical Results

- The truncation error is

$$E_h = \left(\sum_{i=1}^{N-1} (\tilde{u} - u)^2 * h \right)^{1/2}$$

- In this case we compare results using B and B'

h	B	B'
0.20	0.00171780	0.00156968
0.10	0.00038992	0.00034941
0.05	0.00009085	0.00008365

2-D Model Problem: 1st case

- Hyperbolic PDE: Wave equation homogeneous

$$u_{tt} - c^2 div\left(grad\, u \right) = 0 \quad , \qquad x \in [0,1], y \in [0,1]$$

- Initial Conditions:

$$u(0,x,y) = f_1(x,y) = A\sin(kx)\sin(ly)$$
$$u_t(0,x,y) = f_2(x,y) = 0$$

- Exact Solution:

$$u(t,x,y) = A\sin(kx)\sin(ly)\cos(\omega t),$$
$$where\ \ k = m\pi,\ l = n\pi,\ \omega^2 = c^2(k^2 + l^2)$$

- Dirichlet boundary conditions:

$$\alpha\, u = \gamma \quad , \qquad \vec{x} \in \partial\Omega$$

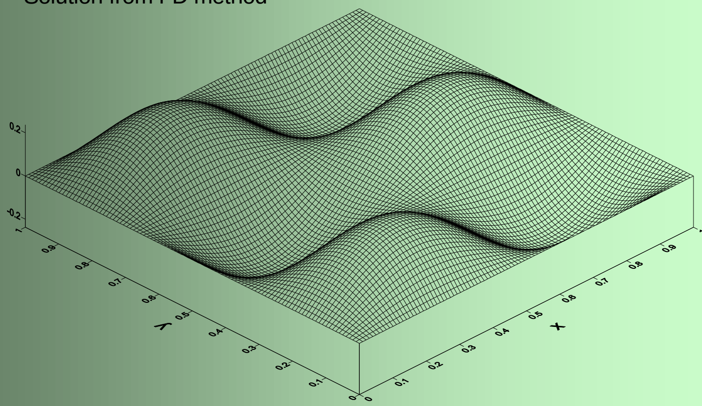
α , γ are functions given on $\partial\Omega$

Numerical results: 1^{st} case

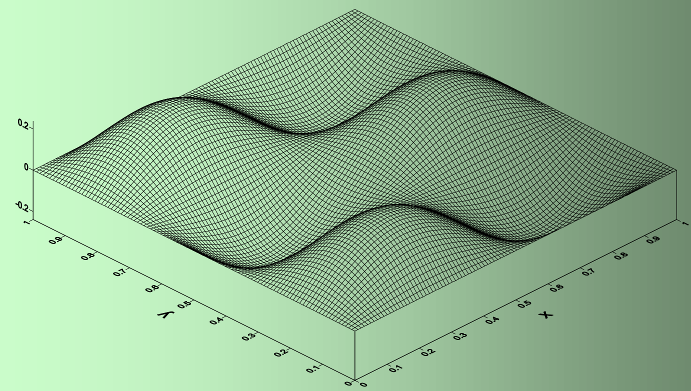
Finite Difference

Mimetic Method

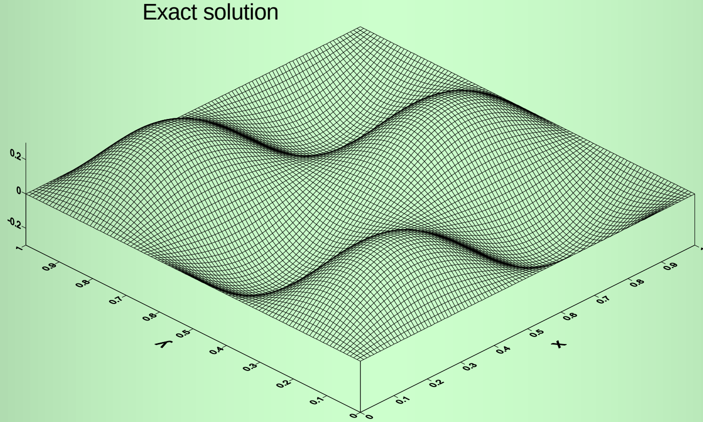
Solution from FD method



Solution from mimetic method



Exact solution



Numerical results

- The truncation error is

$$E_h = \left(\sum_{i=1}^{N-1} (\tilde{u} - u)^2 * h^2 \right)^{1/2}$$

- In this case we compare results using FDM and MM

h	FD	MM
0.20	0.5528	0.4444
0.10	0.2472	0.2063
0.05	0.0828	0.0767
0.025	0.0306	0.0298

Order of Convergence (q)

Truncation Error:

$$E_h \le C\,h^q \quad ; \quad C \;\; convergent \;\; rate \;\; (independent \;\; of \;\; h)$$

Numerical estimation of q:

$$q = \log_2\left(\frac{E_h}{E_{h/2}}\right)$$

h	FD	MM
0.20	-	-
0.10	1.16107	1.1071
0.05	1.5779	1.4274
…	…	…

2-D Model Problem: 2nd case

- Hyperbolic PDE: Wave equation

$$u_{tt} - c^2 div\left(grad\, u \right) = F(\vec{x}) \;\;, \quad \vec{x} \in \Omega$$

F(x) is a source term

- Initial Conditions:

$$u(0,x,y) = f_1(x,y) = 0$$
$$u_t(0,x,y) = f_2(x,y) = 0$$

•Dirichlet boundary conditions:

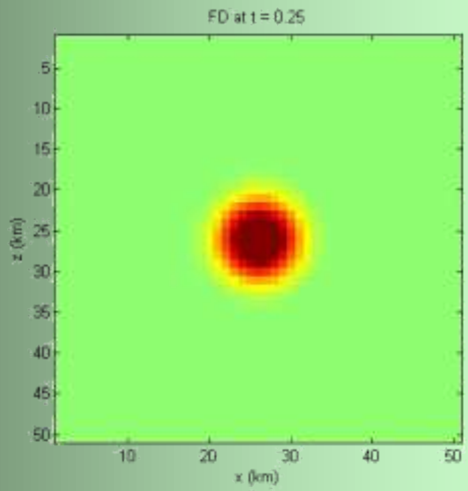
$$\alpha\, u = \gamma \;\;, \quad x \in \partial\Omega$$

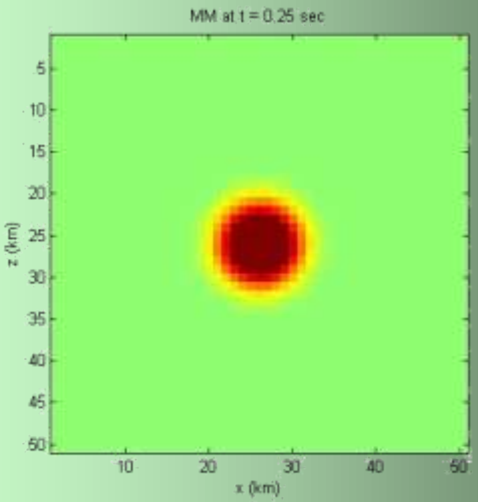
α , γ are functions given on $\partial\Omega$

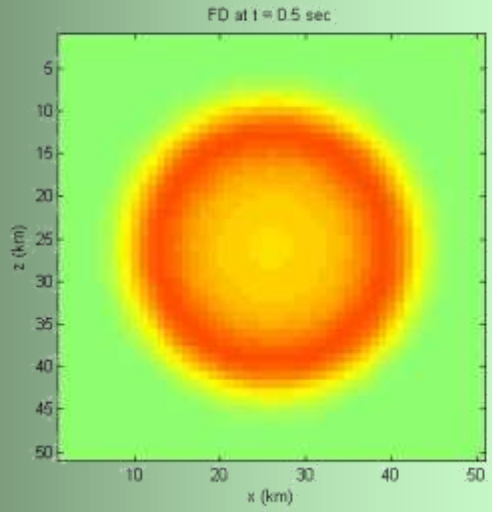
Numerical results: 2nd case

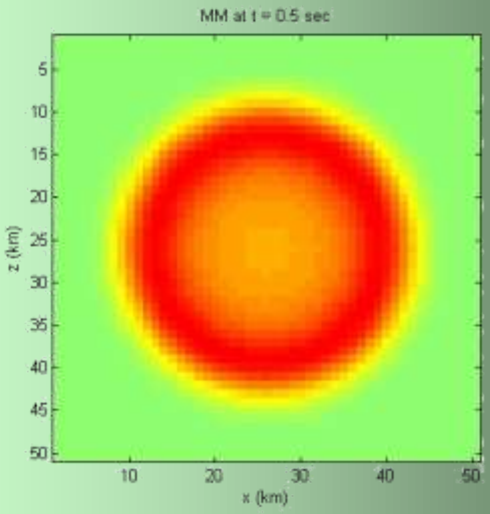
Finite Difference

Mimetic Method









Conclusion and Comments

- Mimetic Methods are simple to implement and reliable method.
- In some cases the numerical experiments shows that mimetic methods is more robust than other algorithms.
- The results are based on the convergence rate, and if results are not better in some cases, it performs equal as the other methods
- It is hoped that this method will be reliable and lead to similar results in higher dimensional cases.

References

- Jose E.Castillo and Mark Yasuda, Linear Systems Arisen for Second Order Mimetic Divergence and Gradient Discretizations. SDSU, San Diego.,3 June 2004, pp. 1-8.

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