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M542 Introduction to Numerical Solutions to Differential Equations

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INTRODUCTION

In this report, we solve 2 problems using the second order mimetic technique developed by J. Castillo, R. Grone and M. Yasuda [1, 2]. This is a method for constructing mimetic discretizations of the gradient and the divergence operators using a matrix formulation to incorporate mimetic constraints.

Some of the most important characteristics of this technique are:

- The orders of the approximations found at the boundary grid points are equal to the interior grid points.
- The method is conservative. That is the mimetic difference approximations satisfy discrete versions of conservation laws and analogies to Stokes' theorem that are true in the continuum and therefore are more likely to produce physically faithful results.

The report is divided in two main parts:

- Firstly, we solve a one dimensional elliptic problem using mimetic discretizations and compare it with the exact analytical solution. We also calculate the error using both a matrix B and a matrix B' (by using weighted inner products).
- Secondly, we solve a two-dimensional linear wave equation, which is an hyperbolic problem using both the classical finite differences and the mimetic technique.

The explicit form of the systems of equations for the one dimensional case and the two dimensional case as well as the Mat lab codes are found in the appendix of this report.

PRELIMINARES

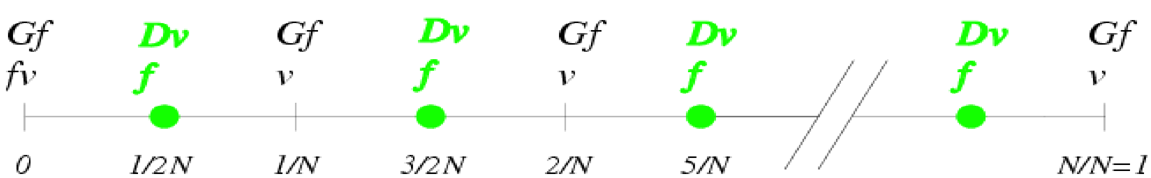
THE MIMETIC OPERATORS

Mimetic methods, as it was mentioned in the Introduction, are a set of discrete operators DIV, GRAD, CURL that approximate continuum differential operators div, grad, curl, and preserve fundamental properties established in vector calculus.

From the Castillo-Grone 2003 paper we have that the central problem is to find DIV, and GRAD to satisfy a discrete analogue of the divergence theorem:

$$\int_{\Omega} div(v)\, f\, dV + \int_{\Omega} v.grad(f)\, dV = \int_{\partial\Omega} f\, v.\hat{n}\, dS$$

Considering the grid shown below:



A collection of values of v and f are defined in a staggered grid and collected in vectors v and f. Operators DIV, and GRAD become matrices D, and G that satisfy

<   v,f > + <v, G f > = < B v, f >

The discrete divergence is defined as

$$(Dv)_{i+\frac{1}{2}} = \frac{v_{i+1} - v_i}{h} \qquad \mathbf{0 \le i \le N\text{-}1}$$

hD = $$\begin{bmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix} \qquad \in R^{N\times(N+1)}$$

And the discrete gradient as:

$$(Gf)_i = \frac{f_{i+1/2} - f_{i-1/2}}{h,} \qquad 0 \le i \le N\text{-}1$$

hG= $$\begin{bmatrix} \frac{-8}{3} & 3 & \frac{-1}{3} & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & 0 & -1 & 1 & 0 \\ 0 & \cdots & 0 & \frac{1}{3} & -3 & \frac{8}{3} \end{bmatrix} \qquad \in R^{(N+1)\times(N+2)}$$

The discrete Laplacian is:

$$L = \hat{D}G$$

$$where \hat{D} = \begin{bmatrix} \mathbf{0} \\ D \\ \mathbf{0} \end{bmatrix} \in R^{(N+2)\times(N+2)} \qquad and \quad A = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \in R^{(N+2)\times(N+2)}$$

Castillo and Grone (2003) introduced matrix B to write the discrete identity using standard inner products,

< D v,f > + <v, G f > = < B v, f >

$$
\mathbf{B=} \quad \begin{bmatrix} -1 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & : \\ : & \dots & \dots & \dots & : \\ : & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}
$$

Therefore: $$B = D + G^T$$

In addition, they proposed matrix *B*' by using weighted inner products,

$$\left\langle \hat{D}v, f_{cb} \right\rangle_Q + \left\langle G^T v, f_{cb} \right\rangle_P = \left\langle B'v, f_{cb} \right\rangle$$

Where Q $\in R^{(N+2)\times(N+2)}$ and P $\in R^{(N+1)\times(N+1)}$ are positive definite matrices consistently built. In this case,

$$B' = D + (PG)^T$$

CASE OF STUDY 1D

In this first case we solve a one dimensional elliptic problem using a uniform staggered grid and a second order mimetic discretizations method. For this case we use Robin boundary conditions, and we calculate 2 approximations one using the matrix B and another using the matrix B', and compare them.

Case 1:

Let, F(x) = $\frac{-\lambda e^{\lambda x}}{e^{\lambda}-1}$

The elliptic partial differential equation is:

$-\nabla^2 u(x) = F(x)$ On [0, 1]

Robin boundary conditions:

$\alpha f(0) - \beta f'(0) = -1$ $\alpha = -e^{\lambda}$

$\alpha f(1) + \beta f'(1) = 0$ $\beta = \frac{(e^{\lambda}-1)}{\lambda}$

And $\lambda = -1$

The exact solution to this problem is:

$$u(x) = \frac{e^{\lambda x}-1}{e^{\lambda}-1}$$

The discretization process leads to these 2 systems of linear equations:

$$(\alpha A + \beta BG + L)\widetilde{u} = f \qquad (1)$$

$$(\alpha A + \beta B' G + L)\widetilde{u} = f \qquad (2)$$

Where f is:

$$f = (-1, F(x_{\frac{1}{2}}), F(x_{\frac{3}{2}}), \ldots\ldots, F(x_{n-\frac{1}{2}}), 0)^T$$

NUMERICAL RESULTS 1-D CASE

In the table 1 we show the truncation error for both approximations and in table 2 we show the order of convergence for the mimetic method using B and B'.

Table 1

1-D TRUNCATION ERROR $$E_h = \left(\sum_{i=1}^{N-1} (\tilde{u} - u)^2 * h \right)^{1/2}$$

h	Error using B	Error using B'
0.20	1.717800E-03	1.569700E-03
0.10	3.899200E-04	3.494100E-04
0.05	9.084900E-05	8.365000E-05

Table 2

ORDER OF CONVERGENCE (q)

Numerical estimation of q: q=log2 (E (h)/E (h/2))

h	q using B	q using B'
0.20	***	***
0.10	2.139312	2.167496
0.05	2.101636	2.062483

CASE OF STUDY 2D

In the second case we solve a 2-dimensional linear wave equation using a nodal grid and finite differences and using a uniform staggered grid and a second order mimetic discretizations method and compare them to the exact solution (two dimensional plane wave solution).

BACKGROUND OF THE PROBLEM

The wave equation is an important partial differential equation which generally describes all kinds of waves, such as sound waves, light waves and water waves. It arises in many different fields, such as acoustics, electromagnetism, and fluid dynamics. Variations of the wave equation are also found in quantum mechanics and general relativity.

The general form of the wave equation for a scalar quantity u is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Where **c** is usually a fixed constant that represents the speed of the wave's propagation.

The basic wave equation is a linear differential equation which means that the amplitude of two waves interacting is simply the sum of the waves.

The one-dimensional form can be derived from considering a flexible string, stretched between two points on a x-axis. It is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

In two dimensions, expanding the Laplacian gives:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

NUMERICAL SOLUTION

Case 2:

The problem to be solve is

$$p_{tt} = c^2 (p_{xx} + p_{yy})$$

Where c = 2500 m/s and a source point (Gaussian function) define by s(t)=exp(-200*(t-0.2)^2) located at the center of the grid.

It was solved using both the mimetic scheme and a 2nd order accurate in time and space finite difference scheme, with the respective space and time steps: dx = dy = 50 m, dt = 0.005 sec, and nt = 400 iterations (2 sec). The grid size is 50 km x 50 km.

$$\frac{p_i^{\ n+1} - 2p_i^{\ n} + p_i^{\ n-1}}{dt^2} = c^2 \left(\frac{p_{i+1}^{\ \ n} - 2p_i^{\ n} + p_{i-1}^{\ \ n}}{dx^2}\right)$$

Initial conditions: $p(0,x) = 0$, $p_t(0,x) = 0$

Rigid Boundary conditions (Dirichlet conditions): p (t, 0) =0, p (t, 50) = 0.

It is important to mention that depending upon the problem different boundary conditions can be used on the edges; approximate-radiation conditions (for simulating an infinite medium), stress free conditions (also known as Neumann condition or free-surface), or zero-velocity conditions equivalent to zero-displacement conditions (Dirichlet condition or rigid surface). For the ease in the studied case we use, as mention before, rigid boundary conditions.

RESULTS

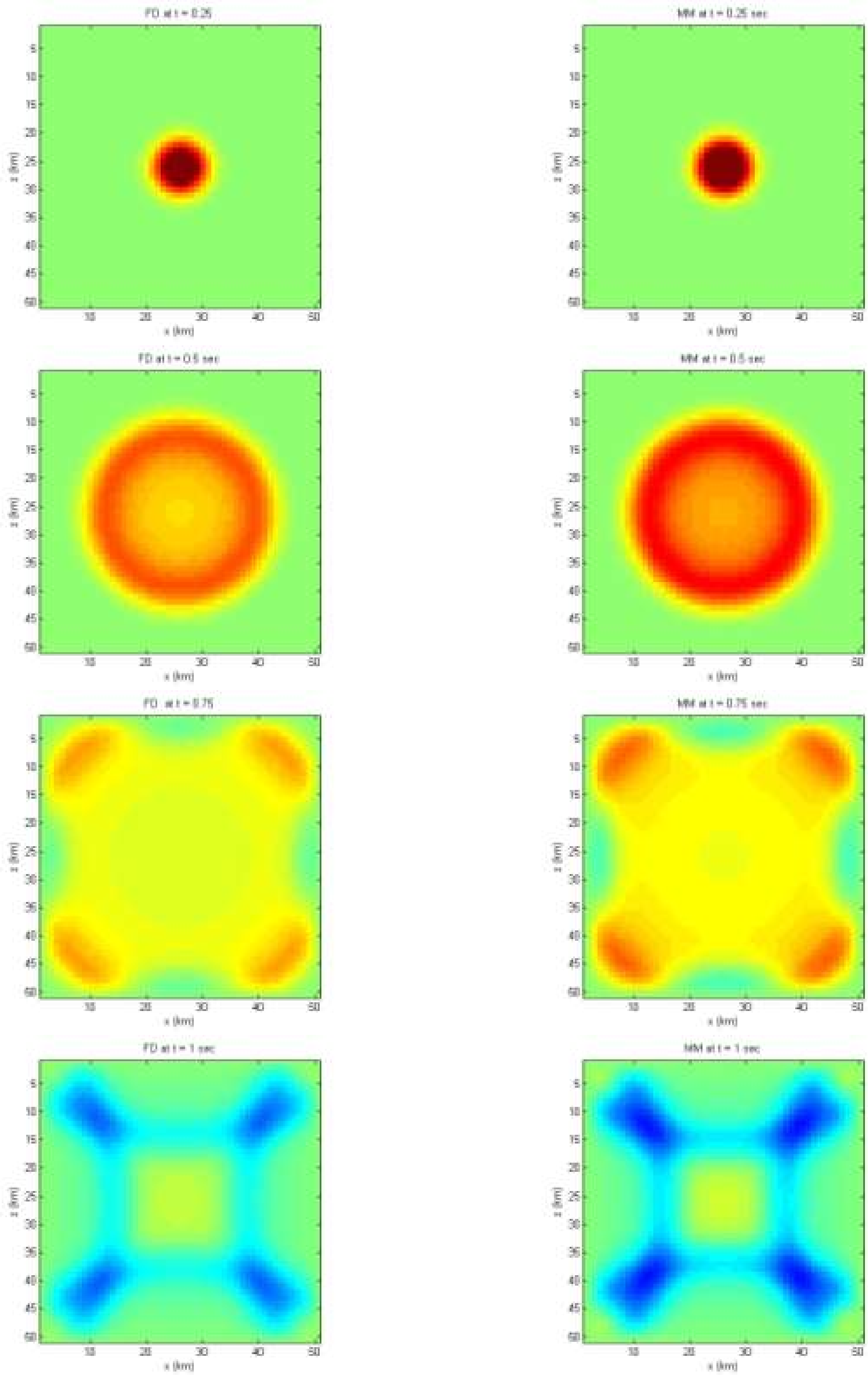


Figure 1-8: Evolution in time of the wave propagation.

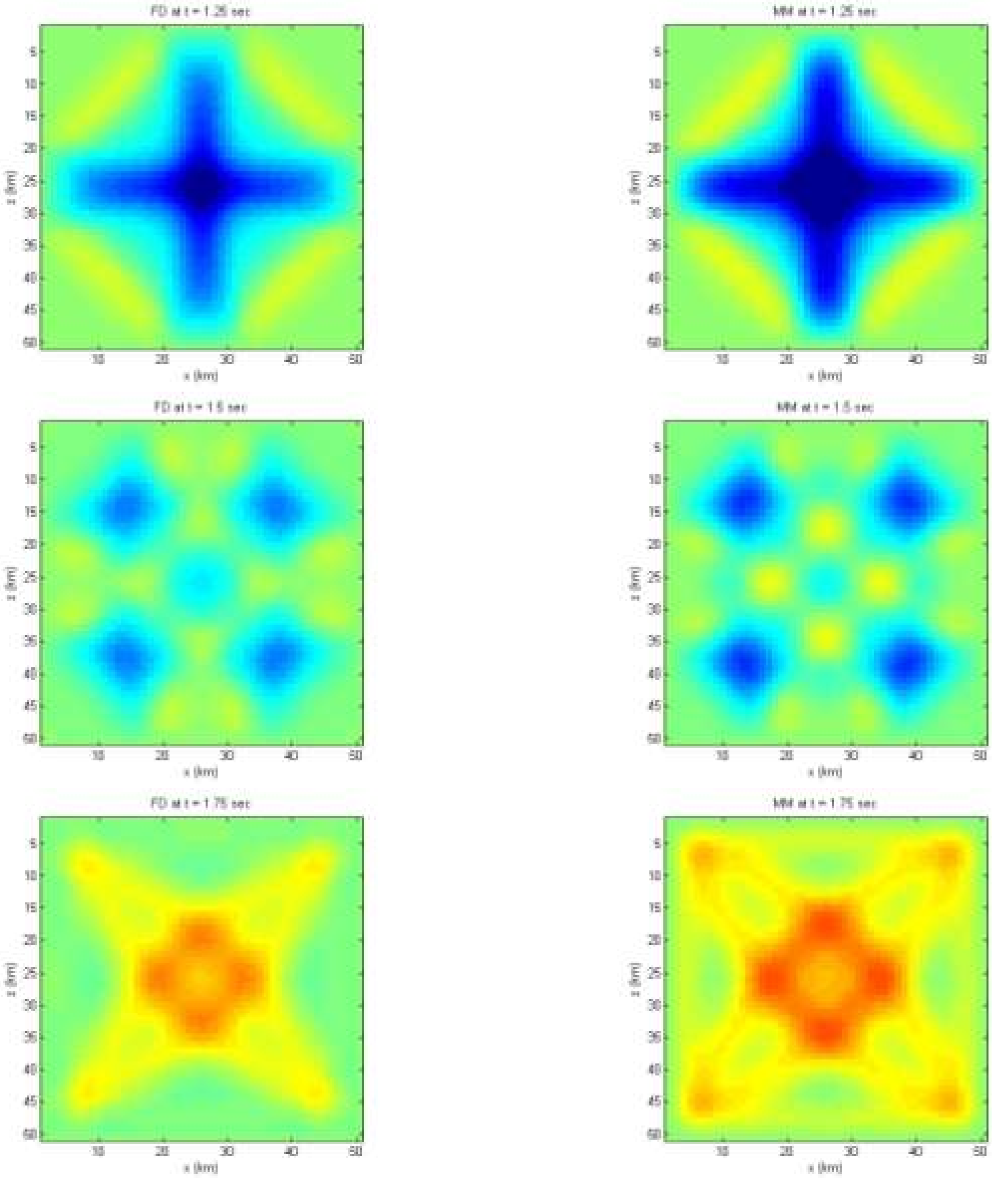
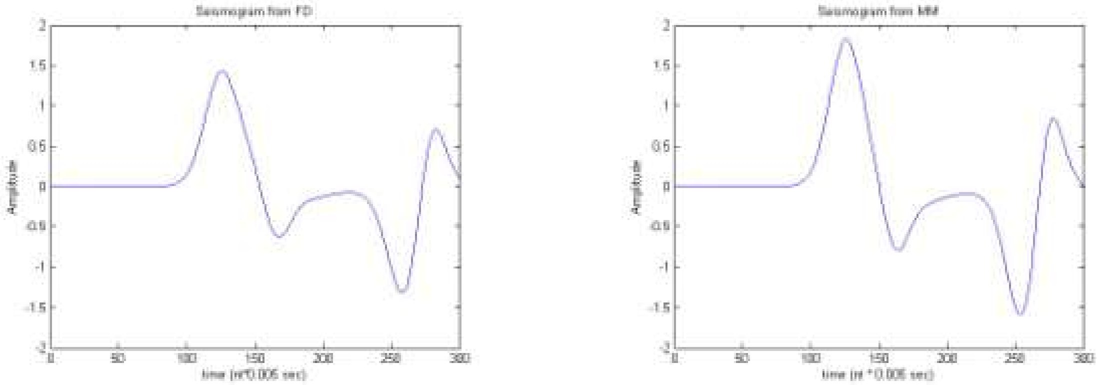


Figure 9-14 Time evolution of the wave propagation



Figures 15-16: Seismograms obtained from both methods in (x, z) = (45,25).

The previous results show a comparison between a method with mimetic operators and a standard finite difference scheme, well established and proved for seismic applications (Matlab code by Olsen K.).

In Figures 1-14 we show the time evolution of the wave in a two-dimensional space and in Figures 15-16 represent seismograms that show that evolution in a specific location.

Unfortunately, we couldn't find an exact solution to compare these two approximations.

CONCLUSION

In this report we present the results of the application of mimetic discretizations in a 1D case for an elliptic problem and show that the mimetic operators improve significantly the performance of the simulation.

We show in the 1D case that the mimetic discretization method is as simple to implement as the standard finite differences. However, the mimetic scheme has a major advantage because is conservative, and do not use any numeric artifice such as ghost points. Moreover, the mimetic scheme is second order on both the boundary and the interior points whereas in the standard FD we drop an order of magnitude in the boundaries.

We also show that the use of the matrix B’ (with weighted inner products) improves the order of convergence.

For the second case we solve a two-dimensional linear wave equation with a Gaussian source point, in a square domain considering initial conditions equal zero and rigid boundary conditions. The problem was solved using a nodal grid and standard finite differences, and a staggered grid and mimetic discretization method.
Unfortunately, we were not able to find a exact solution for this problem therefore we could not show the order of convergence and accuracy of both methods.

REFERENCES

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