Tsunami

Model and Simulation of 1-D Tsunami Utilizing the Non-Linear Shallow Water Wave Equations

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What is a Tsunami?

- Water wave caused by large displacement of water.
- Earthquakes most common cause, which can cause a several meter height displacement over tens of thousands of square kilometers giving huge potential energy to water on surface.
- Waves have wavelengths in the range of many kilometers, with height to length ratios up to 1:100,000
- Wave can increase height at run-up to a steep slope up to 3 or 4 times.

Motivation for Modeling

- Danger to human life requires a warning system.
  "Since 1850 alone, tsunamis have been responsible for the loss of over 420,000 lives and billions of dollars of damage to coastal structures and habitats" (NOAA).
- Only possible if dynamics are known.
  "forecasting tsunami arrival and impact is possible through modeling …" (NOAA)
- Interesting phenomenon to model.

Modeling: What Equations to Use?

- Most tsunami have soliton style dynamics (with some having a “double” wave profile). So why not use KdV?
- Simple reason: From observation, we know we need an equation that includes the depth of the water.
- Solution? Start with the most comprehensive fluid dynamics equation: The Navier-Stokes Equations.

Navier-Stokes Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad (3)$$

The equations require some assumptions.

They are also usually combined with the continuity equation (conservation of mass):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

Can we make it simpler?

- Since Tsunami wave lengths are so long, and heights so small (comparable to the depth of the ocean), they are likened to waves in "shallow" water.
- For this to hold, must have:

$$A/h << 1 \quad \text{and} \quad kh << 1$$

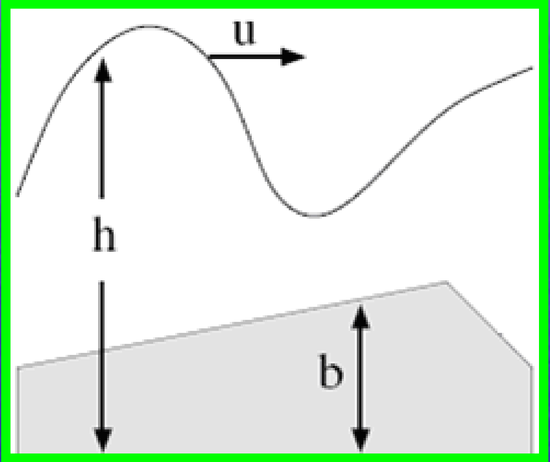
- If we take NS and assume zero viscosity, and reduce to 2-D, and then noting that for a thin layer of a liquid:

$$p = g\,\rho\,h$$

We can obtain an equation for waves traveling on a thin layer of fluid, which are called the Shallow Water Wave Equations:

The 2-D Non-Linear Shallow Water Wave Equations

$$U_t + UU_x + VU_y + gH_x = 0$$
$$V_t + UV_x + VV_y + gH_y = 0$$
$$H_t + [U(H-b)]_x + [V(H-b)]_y = 0$$

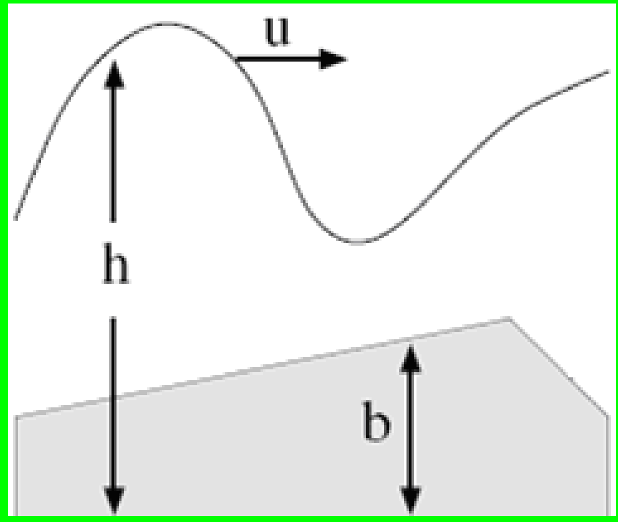


$U(x,t)$ = Horizontal Velocity of $H_2 0$
$V(x,t)$ = Vertical Velocity of $H_2 0$
$H(x,t)$ = Height of the wave
$b(x)$ = Subterrain
g = acceleration due to gravity = 9.8 m/s^2

One-Dimensional Case

If we assume that the velocity in the y-direction is negligible, then the non-linear SWWE reduces to:

$$U_t + UU_x + gH_x = 0$$
$$H_t + [U(H-b)]_x = 0$$



$U(x,t)$ = Horizontal Velocity of H_20
$H(x,t)$ = Height of the wave
$b(x)$ = Subterrain
g = acceleration due to gravity = 9.8 m/s^2

The SWWE – Is it perfect?

There are some limitations to the model:

- Vertical velocity is considered negligible, therefore cannot model some situations such as asteroid impacts, underwater landslides, wave traveling over submerged barrier, or model short-wave-length tsunami.
- Obviously this also means that the equations are inadequate to model the wave crashing over on the beach front, however since tsunami tend to not break at all at run-up, and if they do, do so somewhat inland, this limitation is not such a concern.

Lets Analyze

- Since a tsunami is a wave which travels across distances, we might expect to find a traveling wave solution in the form:

$$H(x,t) = h(z) \quad z = x\text{-}ct$$
$$U(x,t) = u(z) \quad z = x\text{-}ct$$

- However, when one tries (and tries, and tries, and tries and tries ......), you end up not being able to find such a solution, or even a phase-plane analysis form.

- The reason this is so becomes clear on second inspection of the SWWE. Since the wave velocities (thus shape), depends on the depth below it (which is variable), we cannot expect to find a solution valid for all t in which the overall velocity of the wave is constant. For example, if half of the wave is over one depth, and half is over another depth, the total wave's overall velocity will change with time, and thus cannot be a constant.

Tsunami in Open Waters

If Tsunami is far from shore, then the ocean depth does not change much and therefore can take (H-b) to be a constant, d.

Also, as can be seen in numerical simulation, the horizontal velocities do not change much from the initial background velocities, so can replace U with a constant, u.

This gives us:

$$\mathbf{Ut + uUx + gHx = 0}$$

$$\mathbf{Ht + dUx = 0}$$

If we take the background velocities, u, to be 0, then by differentiation, the linear SWWE reduce to:

$$\mathbf{H_{tt} = (gd)H_{xx}}$$

Which is a linear wave equation with a traveling wave solution with velocity dependant on the depth of the ocean: $v = \sqrt{(gh)}$

If ocean is 1km deep, this velocity is ~100m/s ~ 400mph!!

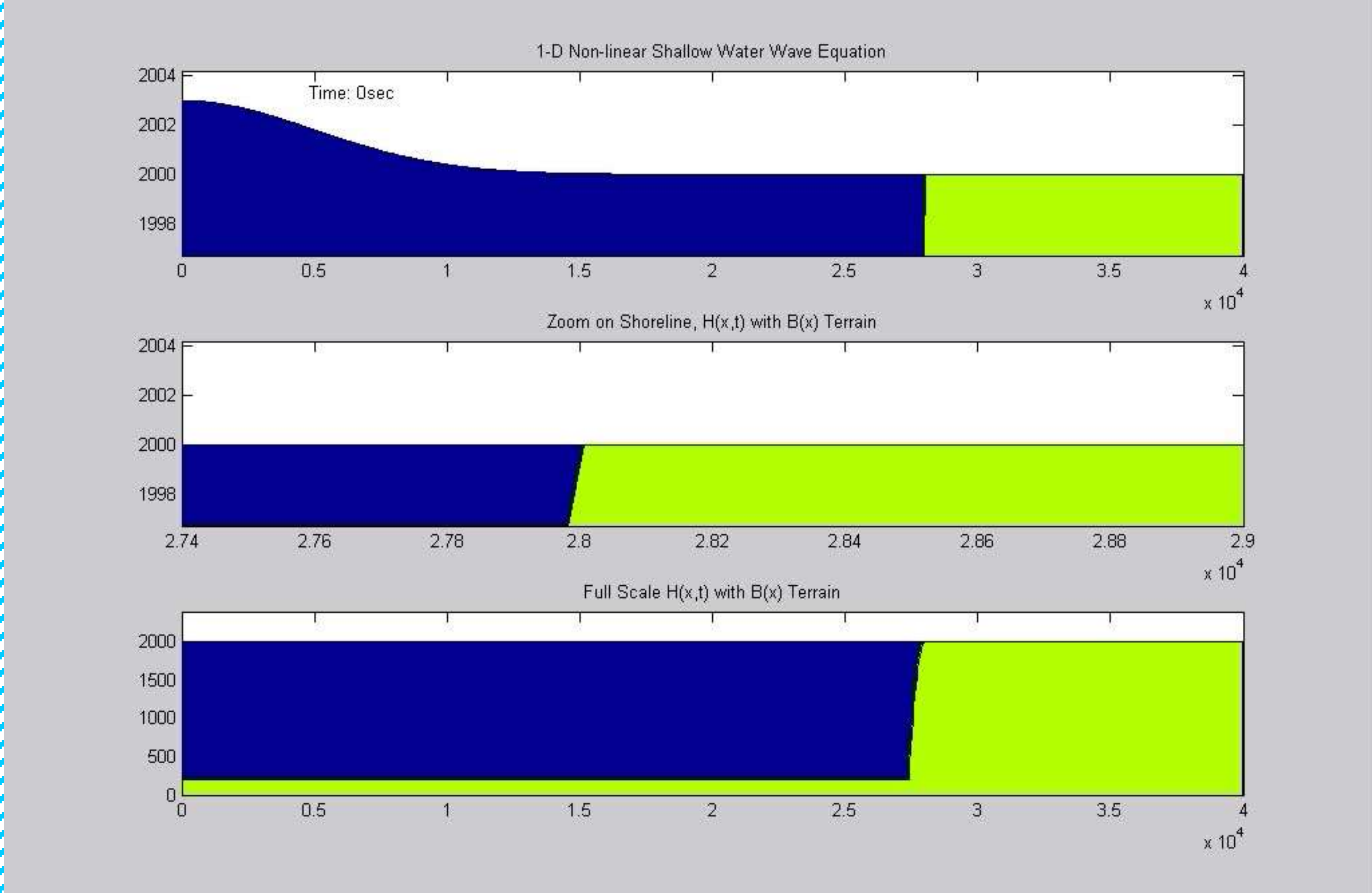
Linear Equation Break Down

- The linear SWWE does simulate a tsunami in open waters but since the waves are so long, as they approach shore, the height of the waves increases too much, and break too early.
- As noted before, Tsunami typically do NOT break at all during run-up, or if they do, much later than the linear theory predicts.
- Thus, linear theory breaks down close to shore.
- Therefore, we use the non-linear SWWE for our simulation.

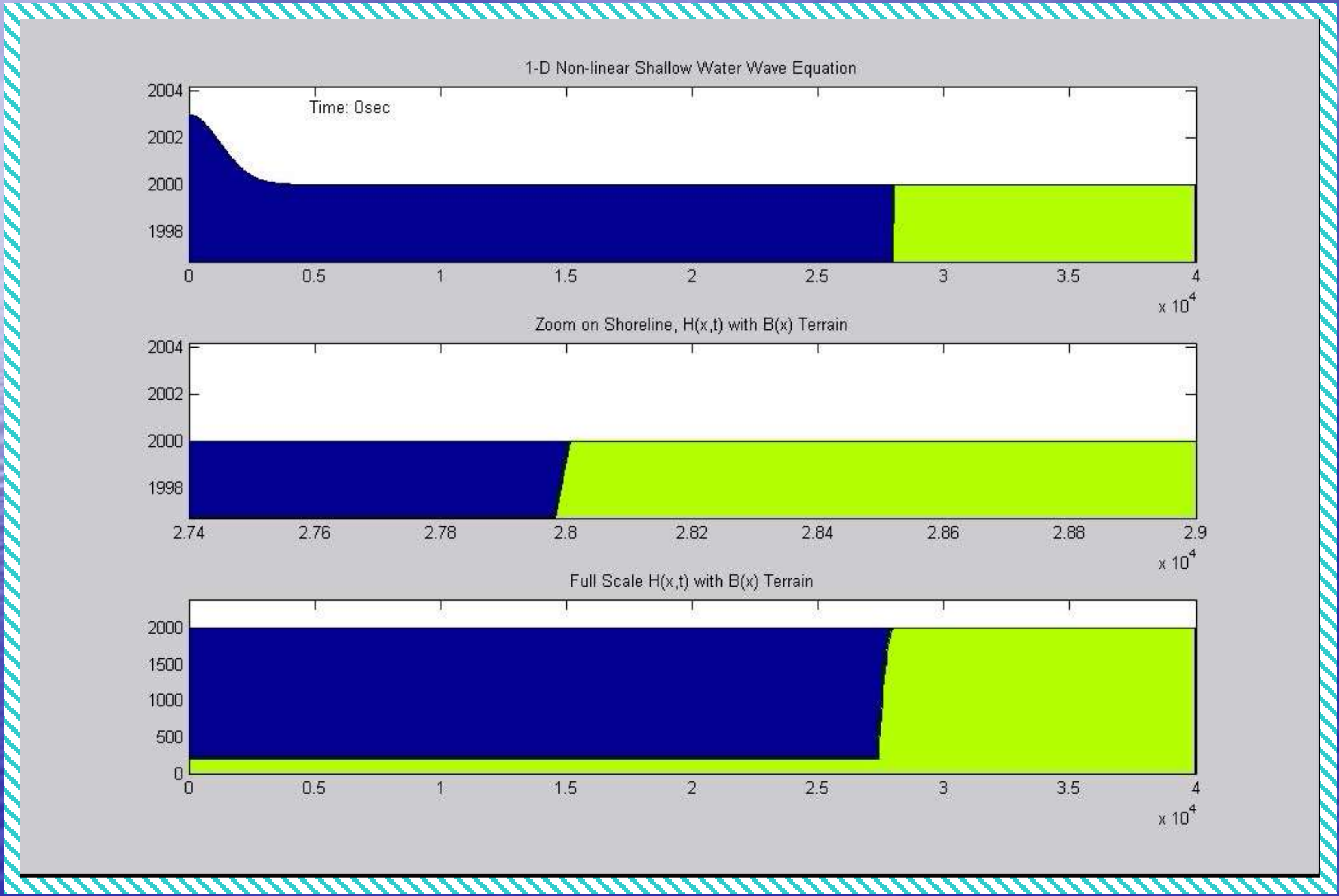
Numerical Analysis

- Non-linear SWWE requires numerical solutions.
- Much research has been done on how to implement efficient numerical schemes, which are quite complicated.
- Simple explicit finite-difference methods cannot be used due to instability.
- To ensure stability, we use a "leap-frog" method.
- Taking a gaussian hump of water 40km long, and 3 meters high, centered 28km off-shore, we obtained the following simulation:

Simulation of 1-D Tsunami



Another run, this time with a 10km initial width



Is this accurate?

What do they say?

"… and can travel great trans-oceanic distances … " (wikipedia.org)

" …reach shore, they slow down… The waves scrunch together like the ribs of an accordion and heave upward. " (PBS)

"… across the ocean at speeds from 500 to 1000 km/h" (wikipedia.org)

"..run-up heights, ..which are greater than the height of the tsunami … by a factor of two or more…" (Bryant, 48)

"After runup, part of the tsunami energy is reflected back to the open ocean. " (USGS)

"Tsunami in most cases do not break, but surge onto shore…" (Bryant, 35)
"(i.e., a rapid, local rise in sea level)" (USGS)

Conclusions

- Despite limitations mentioned previously, the SWWE seems to simulate a non-breaking tsunami very well, both in open waters as well as run-up and coastal flooding.

Any Questions?