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M636 Mathematical Modeling
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FINAL PROJECT

NON LINEAR WAVES:

TSUNAMIS

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INTRODUCTION

After the recent tragic events following the December 2004 Tsunami in the Indian Ocean, it becomes apparent that accurate models of tsunami propagation are required if people are to devise warning systems. Many very realistic and complex models and simulations have been and are being performed by groups across the globe. As an introduction to these concepts and to better understand the problems and techniques involved, we first must review the phenomenon of tsunami.

Tsunami events have been recorded for thousands of years throughout history. They are rare, highly destructive waves which are caused by large displacements of water. Some processes that can create such a displacement are underwater earthquakes, volcanoes, asteroid impacts, extreme weather, and land slides. In these displacements, enormous amounts of water can be displaced, typically a few meters high, and tens of thousands of square kilometers in area. This causes extremely long waves of over 100km wavelengths, propagating at high speeds up to 1000 km/h.

THE MODEL

From the observation of the physical phenomena we know that most tsunami travel long distances without dissipation. One type of wave model that has this behavior is a soliton. Therefore, we try to find models which have non-dispersive solitary solutions. One such model is the KdV equations. However, these equations cannot model a wave approaching a shoreline. Tsunami are observed to have high velocities in open waters, but slow down towards shore. Therefore, the model to simulate them must have a velocity dependant on depth. To find such a model, we start with the most comprehensive fluid dynamics equations and then simplify them to a level that is workable, yet still retains the properties we require.

Since The Navier –Stokes Equations are the foundation of fluid mechanics; they represent the most comprehensive model to describe fluid dynamics. These equations establish that changes in momentum (acceleration) of the particles of a fluid are simply the product of changes in pressure and dissipative viscous forces (similar to friction) acting inside the fluid.

There are basic assumptions necessary to make about the fluid. The first one is that the fluid is continuous. It signifies that it does not contain voids formed, for example, by bubbles of dissolved gases, or that it does not consist of an aggregate of mist-like particles. Another necessary assumption is that all the fields of interest like pressure, velocity, density, temperature, etc., are differentiable (i.e. no phase transitions).

The equations are derived from the basic principles of momentum, and energy and can be written as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad (3)$$

(Where u, v and w are the fluid velocity components in the x, y and z directions, respectively; p denotes pressure; g denotes the gravitational constant; ρ denotes density (assumed constant) and ʋ *denotes the kinematic viscosity of the fluid.)*

The Navier-Stokes equations are usually applied in conjunction with the continuity equation, which states the law of conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

FIRST SIMPLIFICATION

Since Tsunami wave lengths are so long, and heights so small (comparable to the depth of the ocean), the first simplification we can make is that Tsunami can be described as shallow water waves. The condition for this consideration is:

A/h << 1 and kh << 1

Where A is the amplitude of the wave, h is the height and k is the wave number.

This allows us to simplify the NS equations because from fluid mechanics, we know that for a thin layer of liquid, the relation between the pressure and the height is given by the following relationship:

$$p = g\,\rho\,h$$

Where p is pressure, ρ is density and h is height.

With this relation, as well as assuming zero viscosity and considering just 2-dimensional motions, we can reduce the Navier – Stokes equations to:

$$\mathbf{Ut + UUx + VUy + gHx = 0}$$
$$\mathbf{Vt + UVx + VVy + gHy = 0}$$
$$\mathbf{Ht + [U(H\text{-}b)]x + [V(H\text{-}b)]y = 0}$$

U(x,t) = Horizontal Velocity of H20,V(x,t) = Vertical Velocity of H20,H(x,t) = Height of the wave, b(x) = Sub terrain ,and g = acceleration due to gravity = 9.8 m/s2.

Where h is the height field and b is the height of the ground (see fig 1.)

These equations describe the flow of thin layers of fluids and are known as **the shallow water wave equations.**

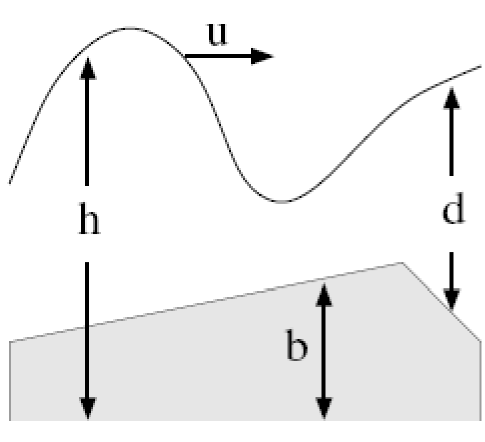

Figure 1: A diagram of a one-dimensional wave.

SECOND SIMPLIFICATION

Since the traveling distance of the tsunami is so large, we can approximate it by a plane wave moving in a single direction. In such a case, we can assume that the velocities of the particles in the perpendicular direction are negligible, and thus such a plane wave can be represented by a one dimensional slice.

Taking this assumption, the non-linear SWWE reduces to:

$$Ut + UUx + gHx = 0$$

$$Ht + [U(H-b)]x = 0$$

Now these equations seem reduce enough to work with. We already mention that Tsunami travel long distances, therefore we now try to find a traveling wave solution of the form:

$$H(x,t) = h(z) \quad z = x-ct$$

$$U(x,t) = f(z) \quad z = x-ct$$

However, we find that after much afford, such solutions do not exist. Therefore let us search for more way to make more approximations.

THIRD SIMPLIFICATION

Since we have a non linear equation the first simplification we should make is to make it linear. In order to do this, let us assume that in the middle of the ocean the depth (H-B) varies very little so we can considered as a constant d. Also, according to [ ] we can assume that the horizontal velocity (U) do not change much therefore we can substitute by a constant (u). Now we have:

$$Ut + uUx + gHx = 0 \qquad (A)$$

$$Ht + dUx = 0 \qquad (B)$$

Let try to plug in a traveling wave solution:

$$-cf' + uf' + gh' = 0$$

$$-ch' + df' = 0$$

Solving for f' in equation B we get:

$$f' = c/(dh')$$

And we plugging this into equation A we get:

$$c = u +- (sqrt(u^2+4dg)/2)$$

This shows that the solution does exist that there is a traveling wave solution with a constant velocity traveling both in the positive and negative direction. However, we do not know what form the solution has.

Returning to our original linear equations notice what happen if we take the background velocities, u, to be 0, the if you differentiate equation A with respect to "x" and equation B with respect to "t", the by combining A and B and collecting terms, the equations reduce to:

$$H_{tt} = (gd)H_{xx} \quad \text{(C)}$$

Which is a standard linear wave equation with a velocity dependant on the depth.

Going back to the value for c obtained using the horizontal velocities u, and replaceing u with 0, yields: $c = \pm\sqrt{dg}$

Also interesting is that multiplying c by itself with a non-zero u value, yields a $c^2 = dg$! Therefore even with a non-zero constant u, the equation is the same as (C).

This result explains why we could not find a traveling wave solution to the non-linear SWWE. The velocity of our linear case is dependant on d, which we take to be constant, thus, a constant velocity. However, in our original SWWE, d was not constant, but changed due to terrain and H(x,t). Therefore, since the velocity of the wave was variable in time, no constant velocity wave solution could possibly be valid for all t.

At this point, we have an approximate solution to a tsunami traveling in open ocean. However according to _________ the linear equation cannot describe accurately the tsunami running up to shore, for it causes the wave's amplitude to rise too high and too fast, thus causing the wave to break too early as compared to observations.

For our simulation, we wish to describe the tsunami as it approaches a shoreline, as well as its formation from the initial disturbance of the water. Therefore, we return to the non-linear form of the SWWE.

LIMITATIONS OF THE SWWE

Before implementing the SWWE, we wish to know its limitations to be sure that it will suit our needs.

In deriving the SWWE, we neglected the z-componants of Navier-Stokes equation, in essence assuming virticle velocities to be negligible. Due to this, the SWWE cannot model certain situations such as asteroid impacts, underwater landslides, wave traveling over submerged barrier, or short-wavelength tsunami.

Also, taking the shallow layer of fluid specification eliminates the possibility of simulating a wave breaking.

These limitations aside, we can see that the SWWE should be adequate enough to model a tsunami caused by a hump of water, traveling to a shoreline and flooding it.

NUMERICAL SIMULATION

To simulate our tsunami, we use matlab to formulate a numeric scheme. The SWWE is inherently unstable when used with standard finite-difference methods. Therefore many in the scientific community have developed complex and efficient implicit schemes for solving the SWWE. These are very difficult to implement, so we instead use a leap-frog explicit scheme which guarantees stability. The major drawback to this method is that it puts strong constraints on step size for t and x.: del-t < del-x/vel.. Since tsunamis cover many hundreds of kilometers, yet beachfronts are in the hundred – meter scale size, this makes full-blown simulations take extreme amounts of time. A possible solution to this would be implementing an adaptive mesh refinement. However for our simulation, we simply use a large del-x and simulate a lower-than-average sized tsunami, formed close to shore.

RESULTS OF SIMULATION

Several characteristics of true tsunamis can be observed in our simulations.

Our tsunami travels un-dispersed over large distances, and slows down as it approaches shore. It travels at speeds similar to real tsunami. When it hits shore, it floods the beachfront in a manner like true tsunami and the wave is reflected back into the ocean. The amplitude of the tsunami doubles in height, yet does not try to break, similar to real tsunami.

CONCLUSION