M638 – Cont. Systems and Chaos

Final Project Report.

Ron Caplan
Rodrigo Negreiros

Chaotic Stellar Dynamo Models

1. INTRODUCTION

Stars in the main sequence (as our Sun) are magnetically active. This magnetic activity depends both on the stars composition and its angular velocity. This activity becomes slower as time passes and the stars rotation speed decreases. Our sun provides a good record of such activity since it has been observed for a long time. These observations include the record of the sunspots and other measurements as the variation of the abundance of certain isotopes as ^{10}Be and ^{14}C. This observation has shown us that the sun magnetic activity has repeated itself irregularly since 1700, with average period of 11 years. It was also observed that the magnetic activity in the sun is modulated in a long-time scale. For instance in the late seventeenth century sunspots almost disappeared completely for almost 70 years.

The aim of the paper studied is to present a simplified model that clarifies the process of modulation of the magnetic activity in stars like the sun. It is assumed that the magnetic field in stars like the sun is generated by a hydrodynamic dynamo located at the convection zone of the star. It is assumed then that the stellar dynamo is a chaotic oscillator.

The authors claim that actual computations using Magneto-Hydrodynamics equations are a formidable task but extremely hard to be performed, and up to this stage has not yet given very satisfactory results. This study seeks then to find a non-linear model which reproduces the behavior observed in sun-like stars and agrees with the present understanding of dynamo theories. This model therefore must have then first, a supercritical Hopf bifurcation where all trajectories are attracted to a periodic solution, secondly it must have a second Hopf bifurcation leading to a quasi-periodic solutions in which the trajectories lies in a two torus, and finally the collapsing of this torus into a chaotic system. This sequence is illustrated in figure 1.

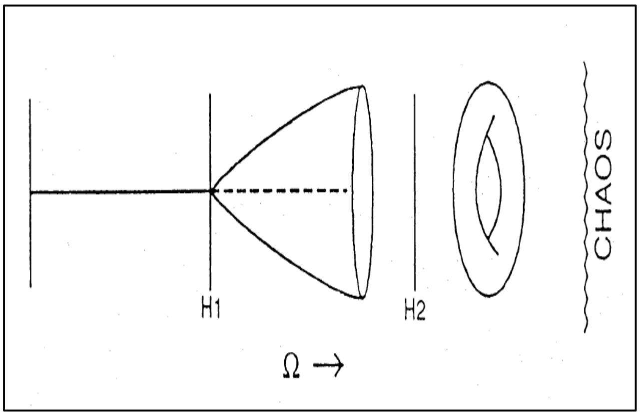


Figure 1

2. FORMULATION OF THE MODEL

2.1 - 1D Model

We start our model simplifying all complicated hydrodynamic information of the system into a single variable *z*. We represent the two different convective states by two fixed points in the *z* axis. Therefore we write our *z* as

$$\dot{z} = \mu - z^2$$

Where the parameter μ is taken as a positive constant. We have then a saddle node bifurcation such that

$$P^+ : z = \sqrt{\mu} \quad \rightarrow \quad Stable$$
$$P^- : z = -\sqrt{\mu} \rightarrow \quad Unstable$$

The other two coordinates of the system will denote the state of the magnetic field. Following the usual notation we separate the field into its Poloidal and Toroidal components. Therefore *y* will be used for the former and *x* for the later.

We identify $r = (x^2 + y^2)^{0.5}$ as the intensity of the magnetic field. We now set

$$q = x + iy = re^{i\phi}$$

We want to reproduce the magnetic instability by having the system going through a supercritical Hopf bifurcation. In order to have this behavior the following equation is adopted

$$\dot{q} = (\lambda + i\omega)q + (a + ib)zq + O(|q|^2 q)$$

Which give us the following system

$$\dot{z} = \mu - z^2$$
$$\dot{x} = (\lambda + az)x - (\omega + bz)y$$
$$\dot{y} = (\lambda + az)y - (\omega + bz)x$$

Converting it to cylindrical coordinates

$$\dot{z} = \mu - z^2$$
$$\dot{r} = \lambda r + azr$$
$$\dot{\phi} = \omega$$

Where in last equations b was set to zero for simplicity.

Due to the fact that the equation on ϕ is uncoupled the system is essentially two dimensional. Therefore doing a stability analysis in the two dimensional system involving *r* and *z* we one would find

	$\lambda < -a\sqrt{\mu}$	$\lambda > -a\sqrt{\mu}$
P^+ :	*Sink*	*Saddle*
	$\lambda < a\sqrt{\mu}$	$\lambda > a\sqrt{\mu}$
P^- :	*Saddle*	*Source*

In order to make the make the model more realistic we need some back reaction from the magnetic field in the flown. This is interaction is done via Lorentz force, which is quadratic in the magnetic field intensity. Therefore we add a term proportional to $|q|^2$. We make sure that we have a negative sign to obtain a supercritical Hopf bifurcation. Our system takes then the following form

$$\dot{z} = \mu - z^2 - (x^2 + y^2) \qquad \dot{z} = \mu - z^2 - r^2$$
$$\dot{x} = (\lambda + az)x - \omega y \qquad \dot{r} = \lambda r + azr$$
$$\dot{y} = (\lambda + az)y - \omega x \qquad \dot{\phi} = \omega$$

Even with last inclusion our system stills two dimensional. Doing a stability analysis we found:

$$|\lambda| < a^2\sqrt{\mu} \rightarrow \Delta > 0$$

$\lambda > 0$ *Unstable*

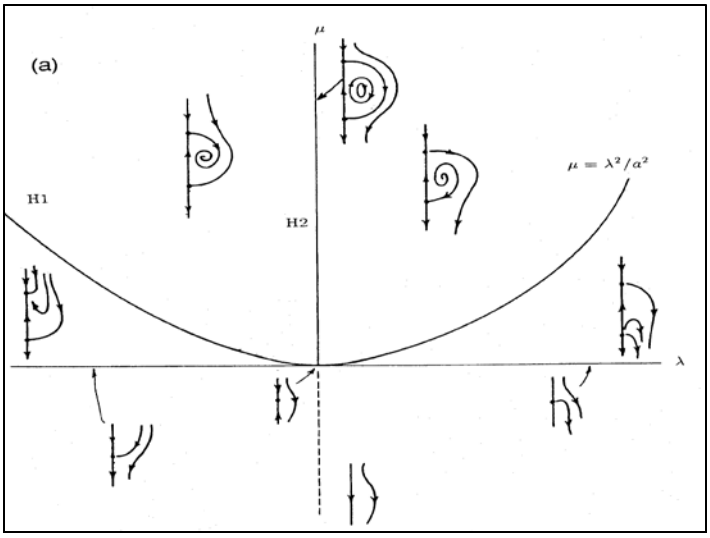
$\lambda = 0$ *Center*

$\lambda < 0$ *Stable*

Spirals

$$-\sqrt{\frac{2a^3\mu}{1+2a}} < \lambda < \sqrt{\frac{2a^3\mu}{1+2a}}$$

The bifurcation diagram for this system is



We reproduced all qualitatively different regions of last diagram using pplane as it was shown in the presentation. We shall not reproduce those figures here to avoid redundancy.

Summarizing, we have so far two Hopf bifurcations. At the first one we have that the fixed point becomes a limit cycle and after the second bifurcation the limit cycles becomes a two-torus. The limit cycles represents the oscillatory dynamo actions whereas the two-torus represents the long-term modulation of the magnetic activity.

Our next step is to break the symmetry that causes the degeneracy. We do that by adding a cubic term on *z* equation. Our system is therefore

$$\dot{z}=\mu-z^2-(x^2+y^2)+cz^3 \qquad \dot{z}=\mu-z^2-r^2+cz^3$$

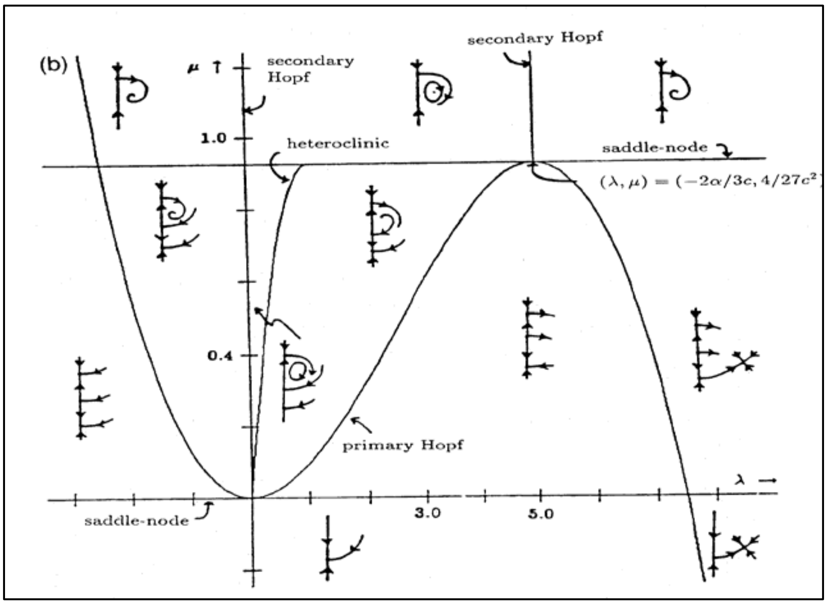
$$\dot{x}=(\lambda+az)x-\omega y \qquad \dot{r}=\lambda r+azr$$

$$\dot{y}=(\lambda+az)y-\omega x \qquad \dot{\phi}=\omega$$

Another stability analysis gives us

$\lambda=0\rightarrow$	$\Delta=2a\mu$	$\mu>0\rightarrow$	*Hopf*
$\lambda=-\frac{2a}{3c}\rightarrow$	$\Delta=2a\mu-\frac{8}{27}\frac{a}{c^2}$	$\mu>\frac{4}{27c^2}\rightarrow$	*Hopf*

Next figure illustrate the parameter space of the system



Although our model can describe a good variety of regimes of magnetic activity we still want the torus to break up into chaos. With the present model this is impossible since the system is still two dimensional. We do that adding a cubic term to the equation that describes the dynamics of the toroidal field. Our system becomes then

$$\begin{aligned} \dot{z} &= \mu - z^2 - (x^2 + y^2) + cz^3 \\ \dot{x} &= (\lambda + az)x - \omega y + dz(x^2 + y^2) \\ \dot{y} &= (\lambda + az)y - \omega x \end{aligned}$$

$$\begin{aligned} \dot{z} &= \mu - z^2 - r^2 + cz^3 \\ \dot{r} &= \lambda r + azr + dr^2 z\cos\phi \\ \dot{\phi} &= \omega - drz\sin\phi \end{aligned}$$

We have now a full 3 dimensional system that might in principle allow chaotic behavior.

3. NUMERIC EXPERIMENTATION

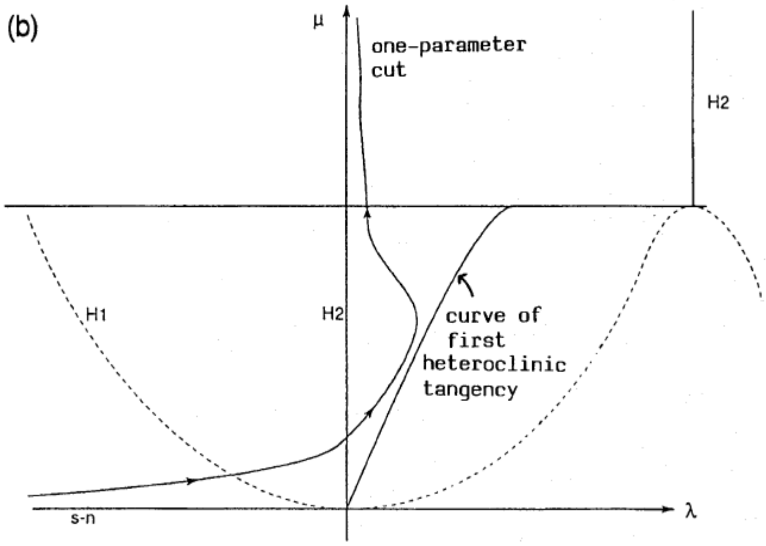
3.1 – Parametric curve in parameter space.

With the objective of simplifying our numerical experimentation we followed the authors steps and chose a parametric curve in the parameter space. The curve is chosen such that it crosses all the interesting regimes in parameter space.

The curve is given by

$$\mu = \Omega^{1/2},$$
$$\lambda = \frac{1}{4}\left[(\ln\Omega + 2)\exp(-\Omega/100)\right]$$

We can see this curve in next figure



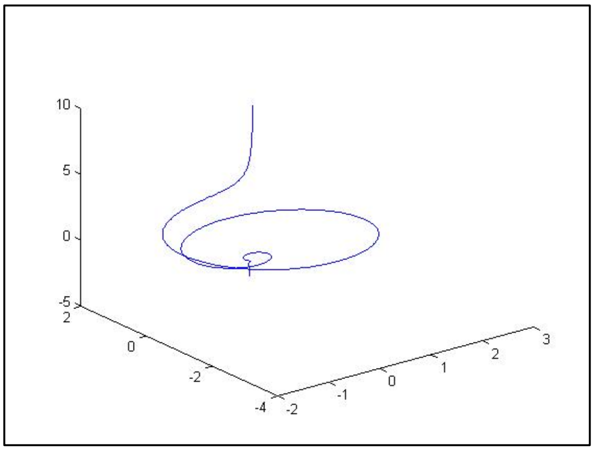
3.2- NUMERICAL RESULTS

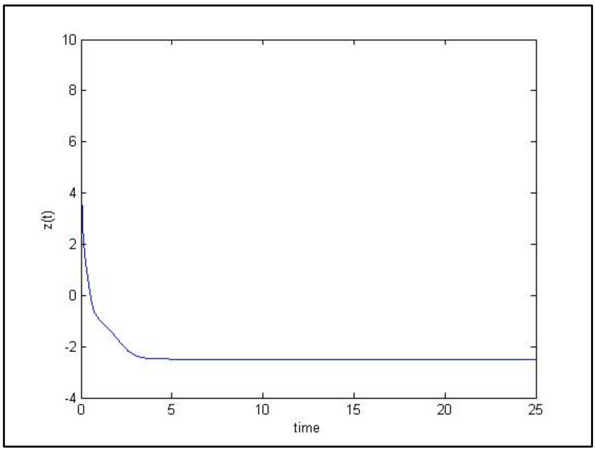
We computed numerical solutions varying the parameter Ω in order to obtain results for all interesting regimes.

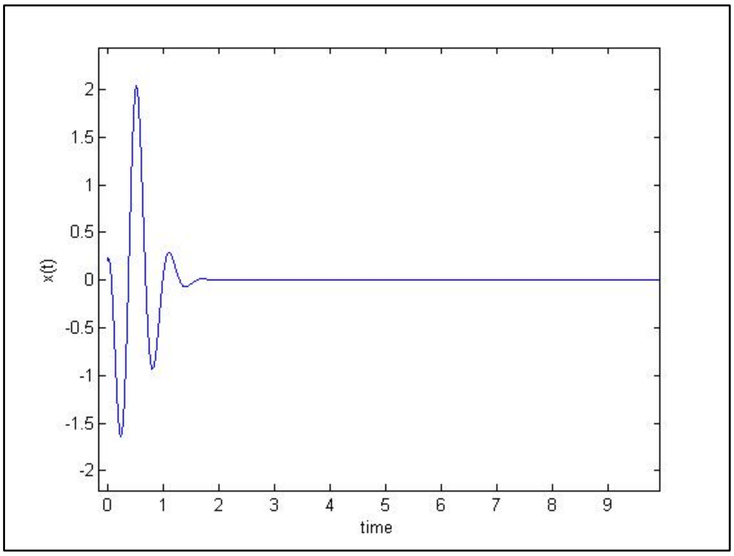
3.2.1 -

$$\Omega << 1$$

In this regime all trajectories collapses towards the fixed point representing the convective state. The results are illustrated on next figures





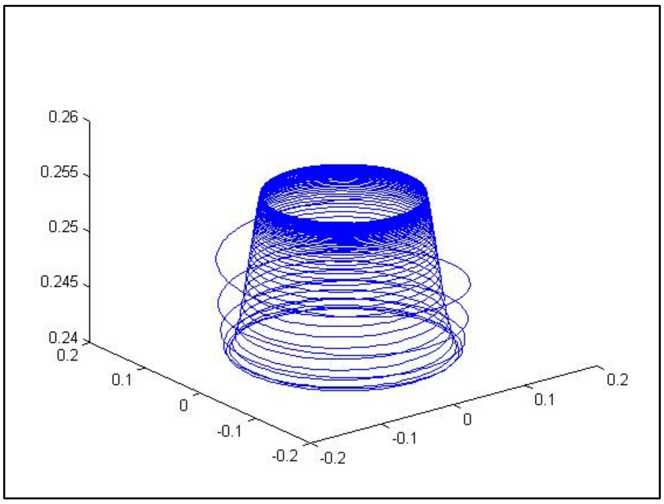


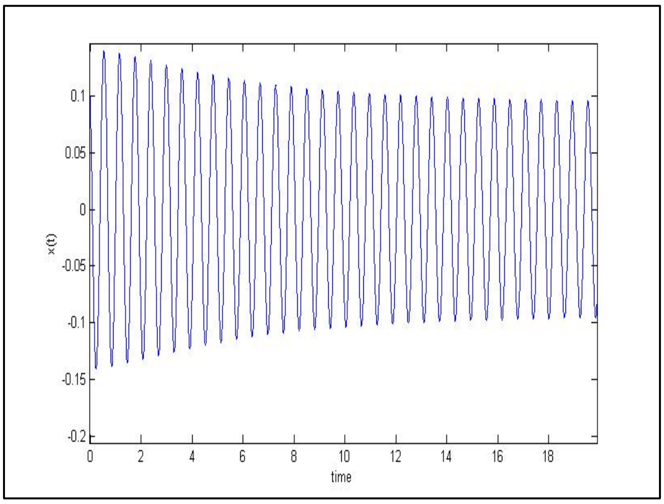
It is evident from last figures that after some small oscillations the system collapses to the fixed point.

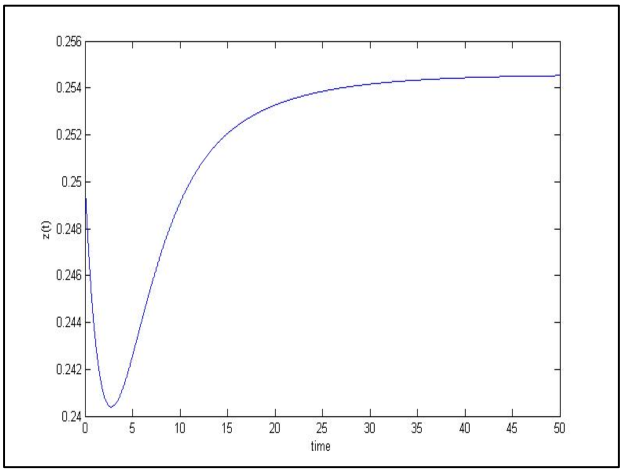
3.2.2 -

$$\Omega = 6.38 \times 10^{-3}$$

At this value our system is slightly above the first Hopf bifurcations. Therefore we should expect a limit cycles.



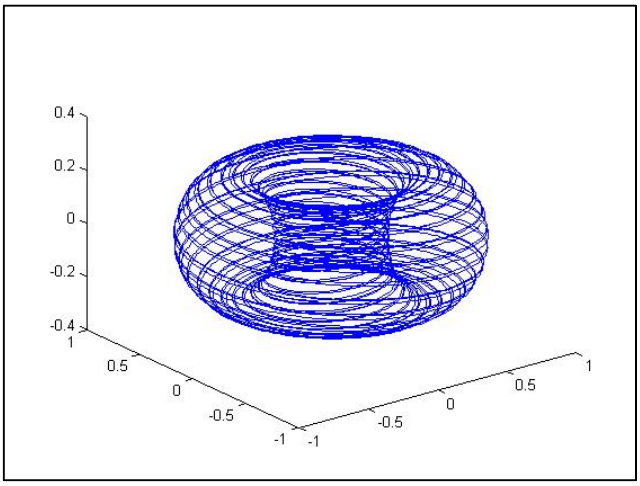


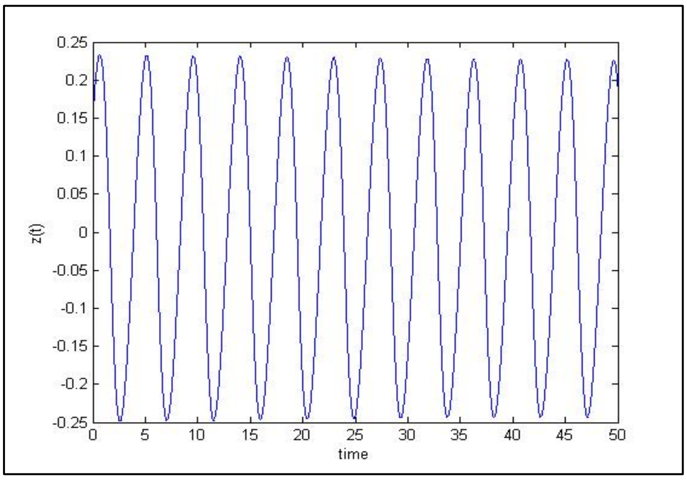


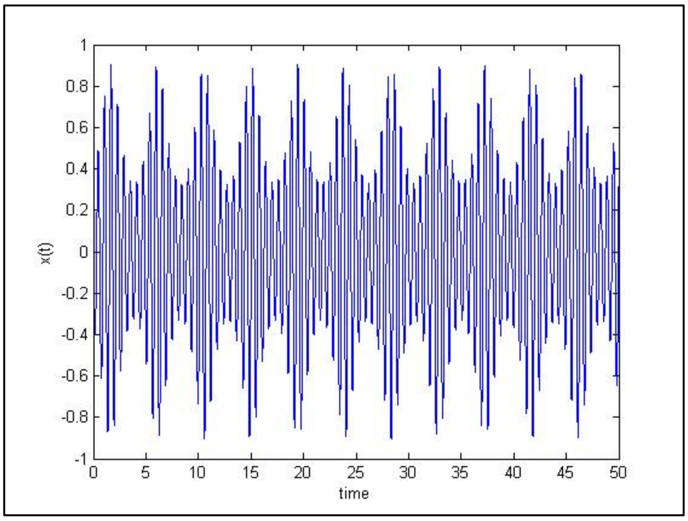
We see clearly that after some transience the system collapses in the limit cycle, where *x* and *y* presents periodic behavior and *z* becomes constant.

3.3 - $\Omega = 1.1 \times e^{-2}$

At this value of Ω the system has gone under the second Hopf bifurcation, therefore we should expect the orbits to be lying onto the two torus.





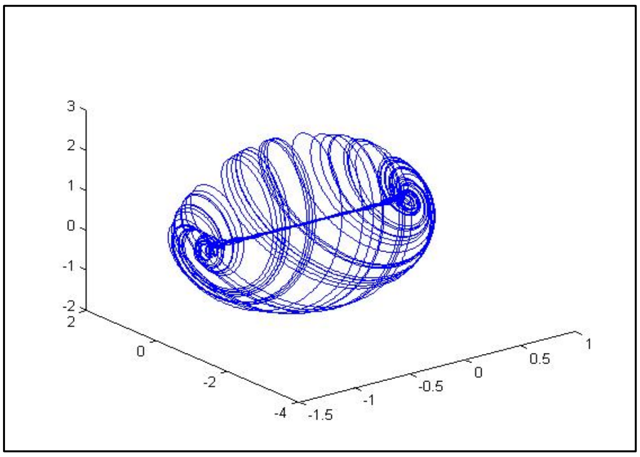


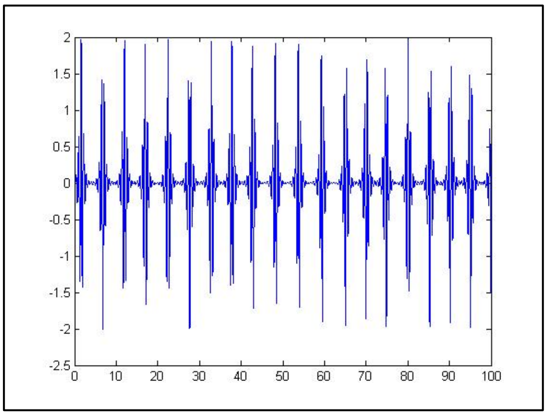
We can see from last pictures that the orbits lie in the surface of the two-torus. Furthermore we see the periodic solutions of *z* and the periodic-modulated solutions for *x* and equivalently for *y*.

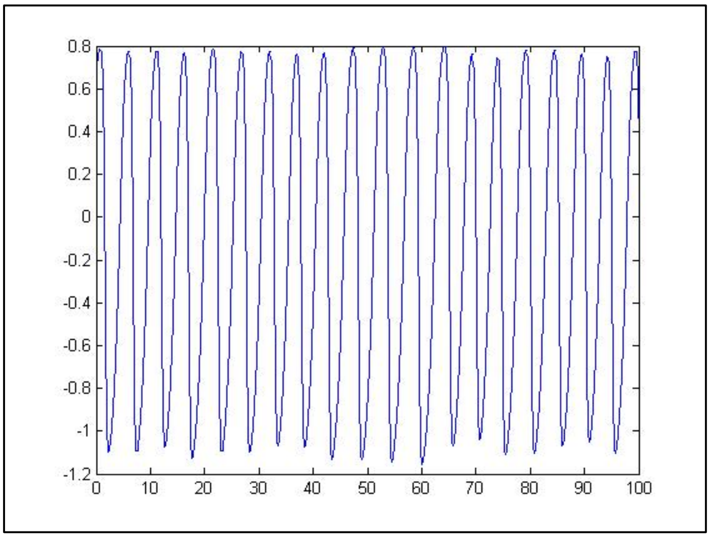
3.4 -

$$\Omega = 0.85$$

We finally reached the regime where the system presents chaotic solutions. As the next figures show







From last picture we can see the aperiodic behavior of system, characteristic of a chaotic regime.

4. CONCLUSIONS

We re-derived the model found in the paper, and in addition we did a detailed analysis of the bifurcations occurring in the system. We could see that the model is fairly successful reproducing the different qualitative regimes of magnetic activity in the star. Even being an artificial model, it might be very helpful to understand the processes occurring in such complex system. Furthermore is a very rich non-linear model in which a great number of features.